International Journal of Theoretical Physics, Vol. 17, No. 7 (1978), pp. 505–542

On Algebras, Manifolds, and Fibre Bundles in Physics[1]

Yutze Chow

Department of Physics, University of Wisconsin, Milwaukee, Wisconsin 53201

Received May 3, 1978

This article surveys the mathematical or physical aspects of algebraic realizations, fibre bundles, supermanifolds, super Lie groups, and super Lie algebras.

This survey concerns some aspects of algebraic approaches to certain physics problems; more specifically we review, comment on, or analyze the following aspects: (i) algebraic realizations, (ii) fibre bundles, (iii) super manifolds, super Lie groups, super Lie algebras. Since (i) and (ii) are older companions than (iii), which is relatively new to physicists, their discussion will be very brief. The third aspect will be our main interest, and it will be mathematical in nature.

1. SOME ASPECTS OF ALGEBRAIC REALIZATIONS

I would like to emphasize that a new way of looking at an old question is often very fruitful. To be definite, I would mention the problem of spin–wave theory in solid-state physics. Take the simple Heisenberg model of a ferromagnet. Holstein and Primakoff [1] were the first ones who attempted to solve it quantum mechanically, leading to the spin–wave concept. Their method was criticized by F. Dyson [2] who attacked the problem with more elaborate techniques. Unfortunately, Dyson's method involves the use of a nonhermitian hamiltonian. The relation between these two different approaches, clouded by details, was hard to see. Yet this situation is easily clarified if we take the viewpoint of algebraic realizations, i.e., if we consider

[1] An expanded and modified version of an invited talk given at the Special Session on Nonassociative Algebras and Their Connections with Physics, at the 749th meeting of the American Mathematical Society, Purdue University, October 29, 1977.

Holstein–Primakoff's approach to be equivalent to the following algebraic realization of spin operators on lattice sites

$$
\begin{aligned}
S_l^{(+)} &= (2s)^{1/2} a_l{}^{+} (1 - a_l{}^{+} a_l / 2s)^{1/2} \\
S_l^{(-)} &= (2s)^{1/2} (1 - a_l{}^{+} a_l / 2s)^{1/2} a_l \\
S_l^{(z)} &= -s + a_l{}^{+} a_l
\end{aligned}
\tag{1.1}
$$

and Dyson's approach to be equivalent to the realization

$$
\begin{aligned}
S_l^{(+)} &= (2s)^{1/2} a_l{}^{+} \\
S_l^{(+)} &= (2s)^{1/2} (1 - a_l{}^{+} a_l / 2s) a_l \\
S_l^{(z)} &= -s + a_l{}^{+} a_l
\end{aligned}
\tag{1.2}
$$

Once we realize this, the relationship between the two approaches becomes very transparent [3]. From this point of view, it is also natural to suggest alternative approaches that may be more convenient or appealing [4]. This is also an example that shows that an algebraic approach does not have to be confined to the area of particle physics (like Weinberg's nonlinear realization [5] in chiral symmetry). Algebraic realizations of groups or algebras of this sort are worthwhile approaches that need to be explored more extensively, at least before a rigorous algebraic quantum field theory becomes computationally practical. A specific realization may not contain all the linear representations, yet the purpose of a realization in this approach is not to arrive at all possible representations but to fix the physical interpretations at the realization level. An interesting mathematical question is the following: we know that for a Lie algebra $\mathscr{L}$ and an $\mathscr{L}$-module V, the second cohomology group $H^2(\mathscr{L}, V)$ corresponds bijectively to the equivalent classes of $\mathscr{L}$-extensions (see, e.g., [6], vol. 2, p. 855) by V. But what can we expect from a *realization* instead of a *representation* under some cohomology theory involving the realization only? This question has never been asked, so far as I know, and would be an interesting one if it turns out to be nontrivial.

2. FIBRE BUNDLES IN PHYSICS

Fibre bundles have been around for quite a while in mathematics, yet their appearance in physics literature was relatively recent. I consider Andrzej Trautman of Warsaw to be one of the earliest campaigners in this direction [7]. It is no mere accident that it takes a general-relativity theorist to get enthusiastic about fibre bundles since they provide very natural (and beautiful) settings for attaching further mathematical structures to a space–time manifold. The mathematical structures may be algebraic in nature or of a group nature, or simply topological in context. So the fibre bundle is a very flexible tool for physicists as well as for mathematicians. It becomes increasingly clear that many problems in physics reveal their clear-cut structures when they are stated in terms of fibre bundles. The best example is the beautiful paper by C. N. Yang on gauge fields [8]. In particular, the theoretical understanding of magnetic monopoles can be channeled so elegantly to the concept of *connections* on a fibre bundle (with $U(1)$ as the structure group). In such a setting, the quantization of a monopole corresponds to the first Chern class of the fibre bundle. It has also become clear in recent years that the transition from a classical dynamic system to quantum mechanical ones can be formulated in terms of fibre bundles. This approach, due to B. Kostant and J-M. Souriau, is usually referred to as *geometric quantization* [9]. We shall give a very brief sketch of this process. Because the phase space of a classical dynamic system is an even-dimensional manifold, a nondegenerate closed two-form (symplectic form) can be introduced to make it a *symplectic manifold.* If M is the configuration space of a classical system, then the cotangent bundle $T^*(M)$ has an even-dimensional bundle space (the phase space) on which we can assign a local coordinate system $\{q^i, p_i\}$. The set $\{q^i\}$ describes the local coordinates on M and $\{p_i\}$ describes the cotangent space, at a point in M, in the sense that every cotangent vector can be written in the form $\sum_{i=1}^{n} p_i \, dq_i$ (where $n \equiv \dim M$). The local coordinate neighborhood in $T^*(M)$ is $\pi^{-1} U$ if its corresponding one in M is U. The symplectic form on $T^*(M)$ can be taken to be, in $\pi^{-1} U$,

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq^i \tag{2.1}$$

where π is the projection from $T^*(M)$ to M. By introducing a globally Hamiltonian vector field ξ_f on $T^*(M)$ for each $f, f' \in C^\infty(M, \mathbb{R})$, we can define the Poisson bracket

$$[f, f']_{\text{PB}} \equiv \omega(\xi_f, \xi_{f'}) \tag{2.2}$$

where

$$\xi_f \equiv \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} \right) \tag{2.3}$$

This makes $C^\infty(M, \mathbb{R})$ a Lie algebra (with respect to the Poisson bracket).

A symplectic form ω, in general, is said to be *integral* if the integration of ω over any closed two-surface (in M) yields an integer. This is also stated sometimes as the condition that the de Rham cohomology class $[\omega] \in H^2(X, \mathbb{R})$ is integral, where X is the symplectic manifold. When this is true, Weil's theorem says that there exists a complex line bundle ζ on M with a Hermitian metric g and a compatible connection ∇ such that the curvature form $\operatorname{curv}(\zeta, \nabla) = \omega$. The space of all sections of ζ, $\operatorname{Sec} \zeta$, forms an infinite-dimensional complex vector space. When equipped with the inner product

$\langle\ ,\ \rangle$ induced by g, the space of all sections is the pre-Hilbert space that yields a Hilbert space $\mathscr{H}$ after completion. For some technical reasons one actually considers, instead of the line bundle $\zeta \equiv \{L, \pi, M, \mathbb{C}, \mathbb{C}^{\natural}\}$, the line bundle $\zeta^{\natural} \equiv \{L^{\natural}, \pi^{\natural}, M, \mathbb{C}^{\natural}, \mathbb{C}^{\natural}\}$ where $\mathbb{C}^{\natural}$ is $\mathbb{C}$ with zero deleted and $L^{\natural}$ is the bundle space whose fibres are $\mathbb{C}^{\natural}$.

Denote by $\tilde{\mathfrak{X}}(L^{\natural})$ the set of all $\mathbb{C}^{\natural}$-invariant real vector fields X on $L^{\natural}$ such that $\pounds_X$ annihilates the canonical one-form of the connection ∇ and X annihilates the real-valued function on $L^{\natural}$ defined by $l \mapsto \langle l, l\rangle$. Further, we denote by $\bar{\mathfrak{X}}(M)$ the set of all globally hamiltonian vector fields on M (i.e., $X \lrcorner\, \omega$ is exact for $X \in \bar{\mathfrak{X}}(M)$). Then the following row-exact commutative diagram which, by the so-called five-lemma, shows that α is a Lie algebra isomorphism between $C^{\infty}(M, \mathbb{R})$ and $\tilde{\mathfrak{X}}(L^{\natural})$. Thus to each classical variable

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \tilde{\mathfrak{X}}(L^{\natural}) & \longrightarrow & \bar{\mathfrak{X}}(M) & \longrightarrow & 0 \\
\| & & \| & & \updownarrow{\scriptstyle\alpha} & & \| & & \| \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & C^{\infty}(M,\mathbb{R}) & \longrightarrow & \bar{\mathfrak{X}}(M) & \longrightarrow & 0
\end{array}$$

$\phi \in C^{\infty}(M, \mathbb{R})$ there corresponds an element $\alpha(\phi)$ that acts as an operator on the Hilbert space $\operatorname{Sec} L$ (after completion). Since α is a Lie algebra isomorphism, the Poisson bracket is mapped into the Lie bracket naturally. Although the actual technical detail is more complex, the use of fibre bundles in this context probes into the nature of the physicist's concept of quantization procedure. It not only questions the precise definition of quantization but also calls for the nontrivial use of some modern mathematical techniques. Cohomology also enters into the picture; for example, equivalent complex line bundles (on the same base manifold) belong to the same class in the first cohomology group (à la Čech) with coefficients in $\mathbb{C}^{\natural}$, and the cohomology classes of symplectic forms ω are just the *Chern classes* of L. It is also worthwhile to note that by Kirillov's orbit method of group representation, geometrical quantization also sheds light on the relation between the irreducible unitary representation of an invariance group of a classical system and the quantization process.

In the context of general relativity, there are many recent efforts in considering nonsymmetric connections [7, 10]. However, it is not generally known that Einstein himself once took an interest in such an approach in 1949; his paper "The Bianchi Identities in Generalized Theory of Gravitation," which appeared in the *Canadian Journal of Mathematics* (1950), contains the following comment that reflects his attitude then:

> The relativistic theory of gravitation bases its field-structure on a symmetric tensor g_{ik}. The most important physical reason for this is that in the special theory we are convinced of the existence

of a "light-cone" ($g_{ik}\, dx^i\, dx^k = 0$) at each world-point, which separates space-like line-elements from time-like ones. What is the most natural way of generalizing this field-structure? The use of a non-symmetric tensor seems to be the simplest possibility, although this cannot be justified convincingly from a physical standpoint. But the following formal reason seems to me important. For the general theory of gravitation it is essential that we can associate with the covariant tensor g_{ik} a contravariant g^{ik}, through the relations $g_{is}g^{ks} = \delta_i{}^k = g_{si}g^{sk}$ (normalized cofactors). This association can be carried over to the non-symmetric case directly. So it is natural to try to extend the theory of gravitation to non-symmetric g_{ik}-fields.

There is, though, an important difference between Einstein's consideration and the so-called Einstein–Cartan theory [7]. Einstein considered a nonsymmetric metric from which a nonsymmetric linear connection is defined. Yet, for a more general consideration one may assume that the metric and the connection are independent of each other. This leads to the extra degree of freedom that was suggested [7, 10] to accommodate the spin of a particle.

Conceptually the various types of connection—linear, Cartan, subordinate structures—can best be understood in a natural setting only in terms of a principal fibre bundle, as formulated by C. Ehresmann [11]. Further possibilities of generalization or specialization of the notion of connection may have to depend largely on the basic structure of a fibre bundle or its generalization. Physics may require these generalizations before they are carried out for purely mathematical interest.

3. INTRODUCTORY REMARKS ON GRADED LIE ALGEBRAS

3.1. Some Notations. Historically, graded Lie algebras were first formulated by Milnor and Moore in 1965 in connection with their study of Hopf algebras [12]. Before we plunge into a full discussion of the "super" objects, let us introduce terminologies and notations. The field **K** is assumed to be not of characteristic 2. Often we restrict the field to $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The vector spaces when graded with respect to $\mathbb{Z}$ will be written $V = \bigoplus_{i\in\mathbb{Z}} V_i$. The term "super" will be used for $\mathbb{Z}_2 \equiv \mathbb{Z}/2\mathbb{Z}$ gradings. Thus a "super" vector space has only two "components," i.e., $V = V_0 \oplus V_1$. We recall that, in general,

$$\text{“degree” of } v \equiv |v| = j \quad \text{if } v \in V_j \qquad \text{(homogeneous)} \tag{3.1}$$

For a graded algebra $A = \bigoplus A_i$,

$$A_k A_j \subset A_{k+j} \tag{3.2}$$

and commutativity requires

$$xy = (-1)^{|x||y|}yx \tag{3.3}$$

for homogeneous elements $x, y \in A$. For inhomogeneous elements, simply consider each homogeneous term separately.

For a *graded Lie algebra* (to be abbreviated GLA) L we have, by (3.2),

$$[L_j, L_k]' \subset L_{j+k} \tag{3.4}$$

where [,]′ denotes *graded Lie multiplication*. The graded version of the antisymmetry relation in GLA is just (3.3), i.e.,

$$[x, y]' = -(-1)^{|x||y|}[y, x]' \tag{3.5}$$

The *graded Jacobi identity* is

$$\sum_{\text{cyc}} (-1)^{|x||z|}[[x, y]', z]' = 0 \tag{3.6}$$

where cyc means a cyclic sum (by permuting x, y, z cyclically). As to "derivations," degrees are assigned to them. By definition, D is a *graded derivation* of degree j if

$$D: A_i \to A_{i+j} \tag{3.7}$$

and

$$D(x \cdot y) = (Dx) \cdot y + (-1)^{|D||x|}x \cdot (Dy) \tag{3.8}$$

An endomorphism also has a "degree" which puts a restriction on the morphism: its restrictions send

$$V_n \to V_{n+j}, \qquad n \in \mathbb{Z} \quad (\text{if } \mathbb{Z}\text{-graded}) \tag{3.9}$$

for an endomorphism of degree j. Denote by $\text{End}^{(j)} V$ the set of all endomorphisms of degree j (on V) and

$$\text{End}^{\#} V \equiv \bigoplus_j \text{End}^{(j)} V \tag{3.10}$$

Naturally, $\text{End}^{\#} V$ can be made into a GLA by defining

$$[x, y]' \equiv x \circ y - (-1)^{|x||y|} y \circ x \tag{3.11}$$

Similarly, one can make any *graded associative algebra* a GLA this way. When $\text{End}^{\#} V$ is considered a GLA with [,]′ defined by (3.11), we denote it by gr V.

A *homomorphism* between GLAs requires grade preserving besides the usual conditions. Thus a representation of (GLA) L is just a GLA-hom from L to gr V, for a given graded vector space V over the same ground field.

3.2. A Physical Example. To avoid confusion, we note that the use of the modifier "super" is different in "supersymmetry" (in physics) and "super Lie algebra" (in mathematical context). In particle physics, the former use accommodates the space–time symmetry (i.e., the inhomogeneous Lorentz group or the spin–statistics symmetry which puts bosons and fermions on the same footing), while the term "super Lie algebra" (or manifold or Lie group) implies $\mathbb{Z}_2$-grading. Hence supersymmetry consideration does not necessarily imply the use of $\mathbb{Z}_2$-grading even if GLA is used. Conversely, there is also no reason to limit the application of SLA to supersymmetry considerations.

Since there are many excellent review articles on applications of SLAs or GLAs to particle physics [13, 14], I shall give only a brief discussion of a simple example, a local quantum field theoretical model in particle physics. This model can be shown to be the same as the minimal spinor extension of the Lie algebra of an inhomogeneous Lorentz group (this extension is an SLA). Consider first the model of three fields: a complex scalar field ϕ (a boson), a two-component spinor field ψ (a fermion), and an auxiliary complex scalar field χ (a boson). Suppose that we now look for a transformation (the so-called supersymmetry group transformation) T that can transform a scalar field (a boson) into a spinor field (a fermion). The simplest relation we can write is this:

$$T_\xi(\phi(x)) = \xi^A\psi_A(x), \qquad A = 1, 2 \tag{3.12}$$

where A is the (two-component) spinor index and ξ^A is a space–time independent anticommuting (in all components) spinor coefficient. We shall also adopt the conventional notation for spinor indices: $\dot{A}$ is attached to the conjugate spinor space. If we assume that the auxiliary scalar field is related to the first-order space–time derivatives of the spinor field through the supersymmetry transformation, then the simplest form is

$$T_\xi(\chi(x)) = -i\bar{\xi}^{\dot{B}}(\sigma_\mu)_{A\dot{B}}\partial^\mu(\psi^A(x)) \tag{3.13}$$

where $\mu = 0, 1, 2, 3$ is the space–time index, $\partial^\mu \equiv \partial/\partial x_\mu$, and $(\sigma_\mu)_{A\dot{B}}$ is defined by the Pauli matrices σ_i for $i = 1, 2, 3$ and $\sigma_0 \equiv 1$. Finally, since the supersymmetry transformation turns the spinor field into scalar fields, the simplest possible relation is

$$T_\xi(\psi_A(x)) = i2\bar{\xi}^{\dot{B}}(\sigma_\mu)_{A\dot{B}}\partial^\mu(\phi(x)) + 2\xi_A\chi(x) \tag{3.14}$$

To find the algebraic properties of the supersymmetry transformation we evaluate

$$(T_\xi \circ T_\zeta)\phi(x) = i2\zeta^A(\sigma_\mu)_{A\dot{B}}\bar{\xi}^{\dot{B}}\partial^\mu\phi(x) + \zeta^A\xi_A\chi(x)$$

which yields

$$[T_\xi, T_\zeta]\phi(x) = i2(\zeta^A(\sigma_\mu)_{A\dot{B}}\bar{\xi}^{\dot{B}} - \xi^A(\sigma_\mu)_{A\dot{B}}\bar{\xi}^{\dot{B}})\partial^\mu\phi(x) \tag{3.15}$$

Furthermore,

$$[T_\xi, \partial^\mu] = 0, \qquad \text{etc.} \tag{3.16}$$

But (3.15) does not have a simple appearance as an algebraic relation. To achieve this, let us write T in terms of some new two-component space–time dependent spinors Q_A and $\bar{Q}_{\dot{B}}$ (as generators of the algebra to be derived):

$$T_\xi \equiv \text{adj}\,\{i(\xi Q + \text{cc})\} \quad \text{when operated on } \phi$$

i.e.,

$$T_\xi(\phi(x)) = i[\xi^A Q_A + \bar{Q}_{\dot{A}}\bar{\xi}^{\dot{A}}, \phi(x)] \tag{3.17}$$

Then we obtain the basic algebraic relations

$$\{Q_A, \bar{Q}_{\dot{B}}\} = 2(\sigma_\mu)_{A\dot{B}}P^\mu \quad \text{(anticommutator)} \tag{3.18}$$

$$\{Q_A, Q_{\dot{B}}\} = 0 \quad \text{(anticommutator)} \tag{3.19}$$

$$[P_\mu, Q_A] = 0 \quad \text{(commutator)} \tag{3.20}$$

where we write P^μ for $-i\partial^\mu$, the generator for space–time translations.

On the other hand, let us consider a possible extension of the Lie algebra of the inhomogeneous Lorentz group L. The generators are defined by

$$[M_{\mu\nu}, M_{\lambda\rho}] = -i(\eta_{\mu\lambda}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\lambda} - \eta_{\mu\rho}M_{\nu\lambda} - \eta_{\nu\lambda}M_{\mu\rho}) \tag{3.21}$$

$$[M_{\mu\nu}, P_\lambda] = -i(\eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu) \tag{3.22}$$

$$[P_\mu, P_\nu] = 0 \tag{3.23}$$

where $\eta_{\mu\lambda}$ is the flat (Minkowski) metric. If we want to construct an extension of L by joining it to a two-component spinor type generator Q_A and its conjugate $\bar{Q}_{\dot{B}}$, we can write the commutators by the definition of a spinor:

$$[M_{\mu\nu}, Q_A] = \tfrac{1}{2}(\sigma_{\mu\nu})_A{}^B Q_B \tag{3.24}$$

where $\sigma_{\mu\nu} = i\frac{1}{2}[\gamma_\mu, \gamma_\nu]$. To close this algebra in the simplest way possible, we put

$$\{Q_A, \bar{Q}_{\dot{B}}\} = 2(\sigma_\mu)_{A\dot{B}}P^\mu \tag{3.25}$$

$$\{Q_A, Q_B\} = 0, \qquad [P_\mu, Q_A] = 0 \tag{3.26}$$

which are seen to be identical to (3.18)–(3.20). This shows that the algebra defined by (3.24)–(3.26), when joined by the Lie algebra of the inhomogeneous Lorentz group, yields an extension which is an SLA, to be denoted by $\mathscr{V}$.

We have a $\mathbb{Z}_2$-grading

$$\mathscr{V} = \mathscr{V}_0 \oplus \mathscr{V}_1 \tag{3.27}$$

with

$$Q_A \in \mathscr{V}_1 \quad \text{and} \quad M_{\mu\nu}, P_\mu \in \mathscr{V}_0 \tag{3.28}$$

Finally, we want to say something about realization. Since in the inhomogeneous Lorentz group P^μ and $M^{\mu\nu}$ can be realized as differential operators in space–time, namely $-i\partial^\mu$ and $i(x^\mu\partial^\nu - x^\nu\partial^\mu)$, one expects by the right-hand side of (3.25) that something can be done about the spinor-type element Q_A of the algebra. It is obviously not possible to do this directly. Hence it is necessary to enlarge the underlying space–time to a larger space V which contains anticommuting "coordinates" (or Grassman algebra) ϑ with

$$\{\vartheta^A, \vartheta^B\} = 0, \qquad A, B = 1, 2 \tag{3.29}$$

In other words, the vector space now becomes a "super" (i.e., $\mathbb{Z}_2$-graded) space:

$$V = V_0 \oplus V_1 \tag{3.30}$$

with $x^\mu \in V_0$ and $\vartheta, \bar{\vartheta} \in V_1$. Therefore the super vector space V is eight-dimensional. On this space we have the realization

$$Q_A = -i\partial/\partial\vartheta^A \tag{3.31}$$

and

$$\bar{Q}_{\dot{A}} = i\partial/\partial\bar{\vartheta}^{\dot{A}} + 2\vartheta^B(\sigma_\mu)_{B\dot{A}}\partial^\mu \tag{3.32}$$

The two Casimir elements in this algebra are the usual $P^\mu P_\mu$ and a Pauli–Lubánski type element $K_{\mu\nu}K^{\mu\nu}$ with

$$K_{\mu\nu} = P_{[\mu}K_{\nu]} \tag{3.33}$$

where

$$K_\nu = \epsilon_{\nu\alpha\beta\mu}M^{\alpha\beta}P^\mu - \tfrac{1}{4}(\sigma_\nu)_{A\dot{B}}[Q^A, \bar{Q}^{\dot{B}}] \tag{3.34}$$

and $\epsilon_{\nu\alpha\beta\mu}$ is the totally antisymmetrical permutation symbol.

3.2. Mathematical Examples. As we know, there are a number of mathematical structures leading naturally to GLAs or, in particular, SLAs. We shall dispense with specific models and mention only a few examples of a general nature.

First we mention the example given at the beginning of Section 3.1, where a *graded vector space* leads to a GLA by considering End$^\#$ V as discussed there. A graded associative algebra also leads naturally to a GLA, as mentioned there.

Next, we give the example of the *Frölicher–Nijenhuis algebra* constructed from a given vector space V. Denote by $A^n(V)$ the vector space of all alternating $(n + 1)$-linear mappings from V^{n+1} to V, and let

$$A(V) \equiv \bigoplus_{n=0}^{\infty} A^n(V) \tag{3.35}$$

Define a product $\bar{\wedge}$ by

$$(f \,\bar{\wedge}\, g)(u_0, \ldots, u_{p+q}) = \sum_\sigma (\operatorname{sgn} \sigma) f(g(u_{\sigma(0)}, \ldots, u_{\sigma(p)}), u_{\sigma(p+1)}, \ldots, u_{\sigma(p+q)}) \tag{3.36}$$

where $f \in A^p(V)$, $g \in A^q(V)$, and $(f \overline{\wedge} g) \in A^{p+q}(V)$ as defined before. σ denotes permutations of $\{0, 1, \ldots, p + q\}$ subject to the conditions $\sigma(0) < \cdots < \sigma(p)$ and $\sigma(p + 1) < \cdots < \sigma(p + q)$. The vector space $A(V)$ then becomes a GLA with respect to the graded Lie product

$$[f, g]' \equiv f \overline{\wedge} g - (-1)^{pq} g \overline{\wedge} f \tag{3.37}$$

It is instructive to consider at this point the case where $f \in A^1(V)$. Then $f\colon V \times V \to V$ (alternating and bilinear) and

$$[f, f]'(u, v, w) = 2\{f(f(u, v), w) + f(f(v, w), u) + f(f(w, u), v)\} \tag{3.38}$$

But char $K \neq 2$, as we assumed in the beginning. Hence (3.38) shows that the Frölicher–Nijenhuis GLA reduces to a Lie algebra if and only if $[f, f]' = 0$.

The third example is the fact that a GLA L, when equipped with a coboundary operator d, passes its "grading" to its *cohomology algebra.* That is, let d be a derivation; then we have

$$d[x, y]' = [dx, y]' + (-1)^{|d||x|}[x, dy]' \tag{3.39}$$

and

$$d \circ d = 0 \tag{3.40}$$

The *p-cocycles* are defined, as usual, by $dx = 0$ and $x \in L_p$. Denote by Z_p the set of all p-cocycles. Then

$$Z \equiv \bigoplus_p Z_p \underset{\text{GLA}}{\subset} L \tag{3.41}$$

Similarly, the p-coboundaries are defined by $x \in L_p$ such that $x \equiv dy$ for some $y \in L_{p-|d|}$. Let B_p be all p-boundaries. Then

$$B \equiv \bigoplus_p B_p \underset{\text{GLA}}{\subset} L \tag{3.42}$$

and in fact B is an ideal of L. Therefore, finally, the p-cohomology group

$$H_p(L) \equiv Z_p/B_p \tag{3.43}$$

yields the cohomology algebra of L (with respect to the coboundary operator d)

$$H(L) \equiv \bigoplus_p H_p(L) \tag{3.44}$$

which is a GLA.

The fourth example we want to mention is a formalism discovered in the study of infinitesimal deformations of complex manifolds.

Denote by z the local complex coordinates. Let ω and γ be, respectively, the vector-valued p-form and q-form with

$$\omega^a \equiv \frac{1}{p!}\sum_j \omega^a{}_{j_1\cdots j_p}\, d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_p} \qquad (a = 1, \ldots, n) \tag{3.47}$$

$$\gamma^a \equiv \frac{1}{q!}\sum_j \gamma^a{}_{k_1\cdots k_q}\, d\bar{z}^{k_1} \wedge \cdots \wedge d\bar{z}^{k_q} \tag{3.48}$$

where $j \equiv \{j_1, \ldots, j_p\}$ and $k \equiv \{k_1, \ldots, k_q\}$. Further, if we let

$$\partial_b\gamma^a \equiv \frac{1}{q!}\sum_k \partial_b\gamma^a{}_{k_1\cdots k_q}\, d\bar{z}^{k_1} \wedge \cdots \wedge d\bar{z}^{k_q} \quad (a, b = 1, \ldots, n) \tag{3.49}$$

$$[\omega, \gamma]' \equiv \sum_{a,b}\{\omega^a \wedge \partial_a\gamma^b - (-1)^{pq}\gamma^a \wedge \partial_a\omega^b\}\partial_b \tag{3.50}$$

then it is obvious that [,]$'$ is bilinear, and it can be shown (by direct calculation) that

$$[\omega, \gamma]' = -(-1)^{pq}[\gamma, \omega]' \tag{3.51}$$

$$(-1)^{pr}[\omega, [\gamma, \rho]']' + (-1)^{qp}[\gamma, [\rho, \omega]']' + (-1)^{rq}[\rho, [\omega, \gamma]']' = 0 \tag{3.52}$$

where ρ is an r-form. Equations (3.51) and (3.52) are just (3.5) and (3.6). In fact, it is also easy to verify

$$\bar{\partial}[\omega, \gamma]' = [\bar{\partial}\omega, \gamma]' + (-1)^p[\omega, \bar{\partial}\gamma]' \tag{3.53}$$

where $\bar{\partial} \equiv \sum_{a=1}^n d\bar{z}^a(\partial/\partial\bar{z}^a)$. That is $\bar{\partial}$ is a graded derivation of degree 1, as defined by (3.8). We mention that if $\omega \in H^1(M, \Theta)$ is an *infinitesimal deformation*, where Θ is the sheaf of sections of the holomorphic $T(M)$, then it can be shown that

$$[\omega, \omega] = 0 \tag{3.54}$$

Our next example concerns the appearance of an SLA structure in a *local Lie algebra* [see A. A. Kirillov, *Uspekhi Mat. Nauk*, **31** (1976), 57].

Consider a real vector bundle $\xi \equiv \{E, \pi, M, F\}$ where the base space M is a manifold and F is a real vector space. Denote by $\Gamma(\xi)$ the set of all smooth sections of ξ. The topology on $\Gamma(\xi)$ is the usual topology of uniform convergence of Γ and its derivatives on a compact set. The set $\Gamma(\xi)$ is said to have a local Lie algebraic structure if $\Gamma(\xi)$ has a (ordinary) Lie algebraic structure such that the Lie multiplication

$$(\chi_1, \chi_2) \mapsto [\chi_1, \chi_2], \qquad \chi_i \in \Gamma(\xi) \tag{3.55}$$

is continuous with respect to both χ_1 and χ_2 and such that

$$\text{supp}\,[\chi_1, \chi_2] \subset \{\text{supp}\,\chi_1 \cap \text{supp}\,\chi_2\} \tag{3.56}$$

where supp means "support of." As a particular example, consider a submanifold M of $\mathbb{R}^{2n}$, with a global coordinate system $x_1, \ldots, x_n, y_1, \ldots, y_n$. Let $\xi \equiv \{M \times \mathbb{R}, \pi, M, \mathbb{R}\}$ be the trivial real line bundle on M. Clearly, $\Gamma(\xi)$ is just $C^\infty(M)$. If we define the Lie multiplication by the Poisson bracket, i.e.,

$$[f, g] \equiv \sum_{i=1}^{n} \{(\partial f/\partial x_i)(\partial g/\partial y_i) - (\partial f/\partial y_i)(\partial g/\partial x_i)\} \tag{3.57}$$

for $f, g \in C^\infty(M)$, then $\Gamma(\xi)$ has a local Lie algebraic structure with respect to the Lie multiplication as defined. For a real line bundle on a manifold M^n, it is obvious that $\Gamma(\xi) = C^\infty(M)$. Take a local coordinate system $x \equiv (x_1, \ldots, x_n)$ on M and write $\partial_i \equiv \partial/\partial x_i$ and also

$$\partial^j \equiv \partial_1^{j_1} \cdots \partial_n^{j_n}, \qquad j \equiv (j_1, \ldots, j_n), j_i \in \mathbb{Z}_+ \tag{3.58}$$

In particular, if $M \subset \mathbb{R}^n$, then it can be shown that every local Lie algebraic structure in $\Gamma(M)$, which is just $C^\infty(M)$ in this case, is defined by

$$[f, g] \equiv \sum_{j,k} h_{jk} \partial_f{}^j \partial_g{}^k \tag{3.59}$$

where $f, g, h_{jk} \in \Gamma(M)$. In fact,

$$h_{jk}(x) = 0 \quad \text{if } |j| > 1 \text{ or } |k| > 1 \tag{3.60}$$

with $|j| \equiv \sum_{i=1}^{n} j_i$. Therefore, (3.59) is effectively

$$[f, g] = \sum_i a^i(x)(f\partial_i g - g\partial_i f) + \sum_{i,j} b^{ij}(x)\partial_i f \partial_j g \tag{3.61}$$

It can further be shown that the vector $\mathbf{a}$ and the bivector $\mathbf{b}$ defined by

$$\mathbf{a} \equiv \sum_i a^i(x)\partial_i \quad \text{and} \quad \mathbf{b} \equiv \sum_{i,j} b^{ij}(x)\partial_i \wedge \partial_j \tag{3.62}$$

together determine a unique local Lie algebraic structure if the following necessary and sufficient conditions are satisfied.

$$\pounds_{\mathbf{a}}(\mathbf{b}) = 0 \qquad \text{(Lie derivative)} \tag{3.63}$$

$$(\delta \mathbf{b}) \wedge \mathbf{b} = \mathbf{a} \wedge \mathbf{b} + \tfrac{1}{2}\delta(\mathbf{b} \wedge \mathbf{b}) \tag{3.64}$$

with

$$\delta \mathbf{c} \equiv \sum_{i_0, i_1, \ldots, i_j} \partial \mathbf{c}^{i_0 i_1 \cdots i_r} \partial_{i_1} \wedge \cdots \wedge \partial_{i_r} \tag{3.65}$$

for any $(r + 1)$-vector $\mathbf{c}$ in general.

An interesting fact is that the following "bracket" operation defined on multivectors yields a GLA structure.

$$[\mathbf{b}, \mathbf{c}]' \equiv \delta(\mathbf{b} \wedge \mathbf{c}) - (\delta \mathbf{b}) \wedge \mathbf{c} - (-1)^{|b|}\mathbf{b} \wedge (\delta \mathbf{c}) \tag{3.66}$$

and the grading is effected by assigning the grade $(r - 1)$ to any r-vector. In the particular case of a bivector $\mathbf{c}$ has grade 1. We note that $[\mathbf{b}, \mathbf{c}]'$ is independent of the local coordinates used, although $\delta\mathbf{c}$ and $\delta\mathbf{b}$ both depend on local coordinates.

Finally, we mention the "Whitehead product" (see, e.g., Hu, *Homotopy Theory*, p. 138) between homotopy groups (at the same point x_0 of a topological space X), which sends

$$[\ ,\ ]' : \pi_m(X, x_0) \times \pi_n(X, x_0) \to \pi_{m+n-1}(X, x_0) \tag{3.67}$$

and satisfies

$$\sum_{\text{cycl}} (-1)^{mp}[[x, y]', z]' = 0 \tag{3.68}$$

where $x \in \pi_m(X, x_0)$, $y \in \pi_n(X, x_0)$, and $z \in \pi_p(X, x_0)$. The symbol $[\ ,\ ]'$ denotes the Whitehead product. Furthermore,

$$[x, y]' = (-1)^{mn}[y, x]' \tag{3.69}$$

However, (3.67) yields a wrong grading, i.e., it is not exactly (3.4). Equation (3.68) is just the required graded Jacobi identity. Yet (3.69) differs from (3.5) by a minus sign. All these can be satisfied by defining a *modified Whitehead product*, to be denoted by $[\ ,\ ]''$:

$$[x, y]'' \equiv (-1)^{m-1}[x, y]' \tag{3.70}$$

Then (3.69) becomes

$$[x, y]'' = -(-1)^{(m-1)(n-1)}[y, x]'' \tag{3.71}$$

and (3.68) becomes

$$\sum_{\text{cycl}} (-1)^{(m-1)(p-1)}[[x, y]'', z]'' = 0 \tag{3.72}$$

From (3.71) and (3.72), it is obvious that by grading π_m in the following way

$$\deg(\pi_m) \equiv m - 1 \tag{3.73}$$

we now have a GLA structure satisfying conditions (3.4)–(3.6). In fact, it is interesting to look into "grading translations." Suppose L satisfies (3.4)–(3.6). But we now change the grading by rewriting

$$L_m \equiv L'_{m+t} \tag{3.74}$$

where t is a fixed integer. Then by (3.4)

$$L'_m L'_n \subset L'_{m+n-t} \tag{3.75}$$

Next we define

$$[x, y]'' \equiv (-1)^{mt}[x, y]' \tag{3.76}$$

for $x \in L'_m$ and $y \in L'_n$. This redefinition of the graded Lie product does not affect (3.75) but changes (3.5) into

$$[x, y]'' = -(-1)^{mn+t}[x, y]'' \tag{3.77}$$

However, the graded Jacobi identity retains its original form, (3.6). This shows clearly that (3.77) becomes (3.5) if t is even, and it becomes

$$[x, y]'' = (-1)^{mn}[x, y]'' \tag{3.78}$$

if t is odd. Equations (3.70)–(3.72) are just the special case where $t = 1$. Notice that in this particular case (3.75) and (3.78) are satisfied by the usual Whitehead product.

An interesting application of the Whitehead product, on homotopy groups, to crystal defects is the discussion by V. Poenaru and G. Toulouse in their preprint, "Topological Solutions and Graded Lie Algebras" (Ecole Normale Supérieure, Paris, 1977).

4. SUPERMANIFOLDS

The fact that a (ordinary) Lie group is a differentiable manifold and the fact that to each (ordinary) finite-dimensional real Lie algebra L there exists a *local Lie group* whose Lie algebra is isomorphic to L suggest that the appearance of GLA indicates the possibility of introducing the notion of supermanifolds and super Lie groups (to be abbreviated SM and SLG, i.e., they are $\mathbb{Z}_2$-graded objects).

4.1. The Berezin–Leĭtes Version of Supermanifolds. The concept of a supermanifold was first introduced by Berezin and Leĭtes in their 1975 paper [16] although the notion of a super Lie group was laid down earlier by Berezin and Kac [17] and to a certain extent by M. Lazard [18]. However, the Berezin–Leĭtes paper gives only a sketchy outline of the theory. The appearance of B. Kostant's most comprehensive work [19] not only brings the theory to a higher degree of rigor and unification but also provides a solid foundation for further development. For this reason, our discussion of Berezin–Leĭtes' work will be very brief.

The key idea is, under the $\mathbb{Z}_2$-grading, to accommodate both commuting and anticommuting elements in an algebra and to relate the algebraic structure to an ordinary differentiable manifold in a certain way. This suggests that either fibre bundles or sheaves may do the job by means of fibres or stalks. Berezin–Leĭtes indicated in their paper [16] that they take the former route, i.e., fibre bundles. Take a manifold M^p (of dimension p); then the commutative associative algebra $C^\infty(U)$ arises for each given coordinate patch U on M^p, where $C^\infty(U)$ is the set of all smooth K-valued functions on U. For each patch U, let there be a correspondence

$$h_U\colon U \to A_U \tag{4.1}$$

where A_U is algebraically isomorphic to $C^\infty(U)$. This yields the even part (i.e., grade 0 in $\mathbb{Z}_2$-grading). The odd part (i.e., grade 1) is obtained by constructing the *exterior algebra* with q generators

$$\{\xi_1, \ldots, \xi_q\} \equiv \xi \tag{4.2}$$

with A_U as the ground ring. Denote this algebra by

$$U_{p,q}(U) \equiv \Lambda^q_{A_U}[\xi] \tag{4.3}$$

To formalize the definition, Berezin–Leĭtes define a **supermanifold** as a pair $\{M^p, \mathfrak{S}_M{}^q\}$ satisfying the following conditions.

1. For each coordinate patch U on M^p, there exists a super K-algebra $\mathfrak{U}_{p,q}(U)$ and a K-homomorphism

$$h_U\colon \mathfrak{U}_{p,q}(U) \to \Lambda^q_{C^\infty(U)}[\xi] \tag{4.4}$$

such that

$$h_U(y_i) = x_i, \qquad h_U(\xi_i) = \xi_i \tag{4.5}$$

with y and ξ generating $\mathfrak{U}_{p,q}$.

2. For any two patches $U \subset V$, there is a grade-preserving epimorphism

$$\rho_{U,V}\colon \mathfrak{U}_{p,q}(U) \twoheadrightarrow \mathfrak{U}_{p,q}(V) \tag{4.6}$$

such that

$$h_U = h_V \circ \rho_{UV} \tag{4.7}$$

3. $\{\mathfrak{U}_{p,q}(U), \rho_{U,V}\}_{U,V} \equiv \mathfrak{S}_M{}^q$ has a fibre bundle structure with base manifold M and fibre space $\mathfrak{U}_{p,q}$.

The Berezin–Leĭtes paper does not make clear whether the formalism is consistent for some technical reasons. Their general approach reveals that it is more adequate to formulate a supermanifold in terms of presheaf or sheaf in the initial step than to take a straight fibre bundle approach.

Table 1 shows the contradistinction between the Berezin–Leĭtes version of supermanifold and a presheaf of K-algebras on a manifold M.

TABLE 1

Berezin–Leĭtes Version	Presheaf of K-Algebras
A coordinate patch U	An open set U
$U \mapsto \mathfrak{U}_{p,q}(U)$	$U \mapsto S_U$
$\rho_{U,V}\colon \mathfrak{U}_{p,q}(U) \twoheadrightarrow \mathfrak{U}_{p,q}(V)$	$\rho_{U,V}\colon S_V \to S_U$
	(restriction mapping)
such that	such that
$h_U = h_V \circ \rho_{UV}$	$\rho_{U,U} = \mathrm{Id}$ (Id $\equiv$ identity mapping)
(hence $\rho_{U,U} = \mathrm{Id}$ and	
$\rho_{V,W} \circ \rho_{U,V} = \rho_{U,W}$	and $\rho_{U,V} \circ \rho_{V,W} = \rho_{UW}$
for patches $U \subset V \subset W$)	for $U \underset{\text{open}}{\subset} V \underset{\text{open}}{\subset} W$

It is important to note that a presheaf is a contravariant functor from the category of open sets (with inclusion maps as morphisms) to the category of abelian groups (with group homs) or, in particular, to the category of K-algebras (with algebra homs). On the other hand, Berezin–Leĭtes formalism indicates the structure of a covariant functor. This is one of the obvious discrepancies.

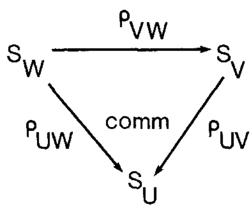
4.2. Prelude to the Kostant Version of Supermanifolds: Canonical Presheaves. Since Kostant formally makes use of the so-called canonical presheaf in defining a supermanifold, we recall a few elementary notions in sheaf theory. Essentially, a presheaf or sheaf assembles local information to global information in a very elegant way. For convenience, we shall specialize in presheaves or sheaves of K-modules.

First, we define a *presheaf*. As we mentioned before, it is essentially a contravariant functor from the category of open sets (and inclusion maps) of a topology space X to the category of K-modules (and K-homs). That is, $\mathscr{S} \equiv \{S_U, \rho_{UV}\}_{U,V}$ is a presheaf of K-modules on X (a paracompact Hausdorff space) if for every open set U of X there corresponds a K-module S_U such that for any open set $V \subset X$ and $U \subset V$ there is a map (the "restriction map")

$$\rho_{UV}\colon S_V \xrightarrow[K\text{-hom}]{} S_U \tag{4.8}$$

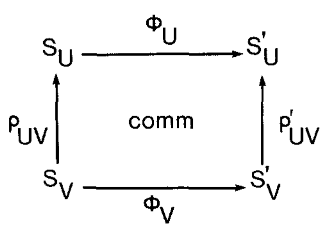
such that, for any open set $W \supset V \supset U$,

$\rho_{UV} \circ \rho_{VW} = \rho_{UW}$ or



(4.9)

and $\rho_{UU} = (\mathrm{Id})_{S_U}$. For two presheaves $\mathscr{S} \equiv \{S_U, \rho_{UV}\}$ and $\mathscr{S}' \equiv \{S'_U, \rho'_{UV}\}$ of K-modules, on X, a **presheaf hom** $\{\phi_U\}_U$ is a collection of K-homs $\phi_U\colon S_U \to S'_U$ such that



(4.10)

An important example of presheaf is given by

$$\eta: U \to C^{\infty}(U) \tag{4.11}$$

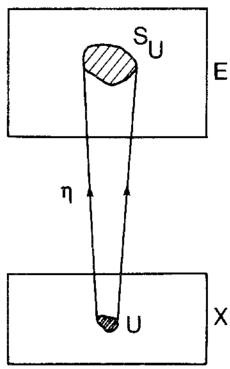
and by

$$\rho_{UV}: C^{\infty}(V) \to C^{\infty}(U) \tag{4.12}$$

defined by

$$\rho_{UV}: F \mapsto f|_U \tag{4.13}$$

A presheaf can be depicted as



(4.14)

The set S_U (perhaps with an algebraic structure and related continuity conditions) will be referred to as the **stalk** over (an open set) U.

In the definition of S_U, in a presheaf, U is assumed to be an open set; S_U is not defined if U is not an open set. Yet, in analogy to a "fibre," it is useful to introduce some object that resembles S_x. To do this, we shall soon introduce the notion of direct limit, which makes it possible for us to define the concept of stalks of a presheaf at a point x.

For convenience, we introduce here the useful notion of a local fibre space. $\xi \equiv \{E, \pi, X\}$ is a **local fibre space** over X if π (the *projection map*) is a local homeomorphism as well as a (globally) continuous surjection from topological spaces E onto X. Each fibre $\pi^{-1}x$ is referred to as the **stalk** of ξ at the point $x \in X$. A **sheaf** ξ of K-modules over X is defined as a local fibre space $\{E, \pi, X\} \equiv \xi$ such that each stalk $\pi^{-1}x$ is a K-module whose operations (addition and scalar multiplication) are continuous on E. Similarly, for a sheaf of K-algebras, the algebraic operations must be continuous. The *sections* of a local fibre space, or of a sheaf, are defined in the usual way.

The correspondence between a sheaf and a presheaf is established through the concept of an *equivalent relation.* Let $\{S_U, \rho_{UV}\} \equiv \mathcal{S}$ be a presheaf of K-modules on X. Construct the disjoint union,

$$\mathcal{S}_x^{\ddagger} \equiv \bigcup_{U \in N_x} S_U, \qquad N_x \equiv \text{all open sets of } x \tag{4.15}$$

An *equivalence* relation $\approx$ on $\mathscr{S}_x^\sharp$ is defined, for $a \in S_U$ and $b \in S_V$ with $U, V \in N_x$, by

$$a \approx b \quad \text{iff} \quad \exists W \in N_x \text{ such that } W \subset U \cap V \quad \text{and} \quad \rho_{WU}a = \rho_{WV}b \tag{4.16}$$

Denote the *equivalent class* of a by $[a]$. Then $[a]$ is called the germ of a at x. The collection of all equivalent classes in $\mathscr{S}_x^\sharp$, for a given point $x \in X$, is called the **direct limit** of $\mathscr{S}_x^\sharp$ when considered a K-module.

Symbolically, denote

$$\mathscr{S}_x \equiv \{[a] | a \in \mathscr{S}_x^\sharp\} \equiv \varinjlim_{x \in U} S_U \quad \text{(conventional notation)} \tag{4.17}$$

The K-module structure of $\mathscr{S}_x$ is defined by means of the mapping

$$\rho_{x,U}: S_U \to \mathscr{S}_x \quad \text{with } \rho_{x,U}: a \mapsto [a] \tag{4.18}$$

Addition in $\mathscr{S}_x$ is defined by

$$[a] + [b] \equiv \rho_{x,W}(\rho_{WU}a + \rho_{WV}b) \tag{4.19}$$

and *scalar multiplication* by

$$\kappa[a] \equiv \rho_{x,U}(\kappa a) \tag{4.20}$$

Construct now

$$\mathscr{S}^\sharp \equiv \bigcup_{x \in X} \mathscr{S}_x \tag{4.21}$$

Then $\mathscr{S}^\sharp$ can be endowed with the structure of a local fibre space by first defining the projection map $\pi: \mathscr{S}^\sharp \to X$ by

$$\pi: \mathscr{S}_x \mapsto x \tag{4.22}$$

Next, we topologize the space $\mathscr{S}^\sharp$ by using the collection of all sets of the form

$$E_a \equiv \{\rho_{x,U}a | x \in U\}, \qquad a \in S_U \ (U \text{ open in } X) \tag{4.23}$$

as a basis of a topology. It is easy to see that π is a *local homeomorphism* as well as (globally) continuous. Besides, each stalk $\mathscr{S}_x$ is a K-module, and the mappings $([a], [b]) \mapsto [a] - [b]$ and $[a] \mapsto \kappa[a]$ are both continuous. Thus, $\mathscr{S}^\sharp$ is indeed a sheaf of K-modules on X, to be called the **sheaf of germs** of $\mathscr{S}$.

We now define a canonical presheaf. Let $\mathscr{S} \equiv \{S_U, \rho_{UV}\}_{U,V}$ be a presheaf on X. $\mathscr{S}$ is a **canonical presheaf** if, for any open cover $\{U_i\}_I$ of an open set U (of X), the following conditions are satisfied.

1. If $a, b \in S_U$ and $\rho_{U_iU}a = \rho_{U_iU}b$ for all $i \in I$, then

$$a = b \tag{4.24}$$

2. If $\{a_i\}_I$ is a set of elements such that for $a_i \in S_{U_i}$

$$\rho_{U_{ij}U_i}a_i = \rho_{U_{ij}U_j}a_j \qquad (U_{ij} \equiv U_i \cap U_j \neq \varnothing) \tag{4.25}$$

for all $i, j \in I$, then $\exists b \in S_U$ such that

$$\rho_{U_i U} a = a_i, \quad \text{all } i \in I \tag{4.26}$$

An important example of a canonical presheaf is (4.11), the canonical presheaf of all smooth $\mathbb{R}$-valued functions on X (or all holomorphic $\mathbb{C}$-valued functions). Another important example is the following: given a continuous surjection π from topological spaces E onto X, the collection of all local sections forms a canonical presheaf.

For any U open in X, denote the set of all sections over U by

$$\Gamma_U \equiv \{\chi | \text{continuous maps } \chi\colon U \to E, \pi \circ \chi = \mathrm{Id}_U\} \tag{4.27}$$

and for $U \subset V$ open in X, define the restriction maps by

$$\rho_{UV}\colon \Gamma_V \to \Gamma_U \quad \text{with } \chi \mapsto \chi|_U \tag{4.28}$$

It is easy to check that conditions (4.24) and (4.25) are satisfied.

In particular, we take a presheaf $\mathscr{S}$ (of sets, or with possibly some algebraic structures) on X. Then $\Gamma(\mathscr{S}^\sharp)$, the set of all sections of $\mathscr{S}^\sharp$ (the sheaf of germs of $\mathscr{S}$), is a canonical presheaf. To be more precise, $\Gamma(\mathscr{S}^\sharp) \equiv \{\Gamma(U, \mathscr{S}^\sharp), \rho_{UV}\}$ is defined by

$$\eta\colon U \mapsto \Gamma_U \tag{4.29}$$

and

$$\rho_{UV}\colon \Gamma_V \to \Gamma_U \quad (\text{sending } \chi \mapsto \chi|_U) \tag{4.30}$$

for $U \subset V$ both open in X. $\Gamma(\mathscr{S}^\sharp)$ is often called the **presheaf of sections** of the sheaf of germs of $\mathscr{S}$. We note that this situation is actually of a general nature; $\Gamma(\mathfrak{S})$ can be defined clearly for any sheaf or canonical sheaf $\mathfrak{S}$ instead of $\mathscr{S}^\sharp$. In particular, we are interested in the case of sheaves or canonical sheaves of K-modules or K-algebras. In such a case, we need algebraic operations on sections. Addition of sections is defined by

$$(\chi + \chi')x = \chi(x) + \chi'(x), \qquad x \in U \tag{4.31}$$

and *scalar multiplication* by

$$(\kappa\chi)x = \kappa(\chi(x)) \tag{4.32}$$

and (for K-algebraic cases) the algebraic operation by

$$(\chi \cdot \chi')x = \chi(x) \cdot \chi'(x) \tag{4.33}$$

It can be shown that, for any canonical presheaf $\mathscr{S}$, the presheaf mapping

$$\gamma\colon \mathscr{S} \to \Gamma(\mathscr{S}^\sharp) \tag{4.34}$$

sending

$$S_U \mapsto \Gamma_U \tag{4.35}$$

for any U open in X, is a bijection. In particular, if $\mathscr{S}$ is a presheaf of abelian groups, then γ is a *presheaf isomorphism.* As $\Gamma(\mathscr{S}^\#)$ is a canonical presheaf, (4.34) says that to every canonical presheaf $\mathscr{S}$ one can associate a local fibre space whose canonical presheaf of sections is just the original $\mathscr{S}$. Equation (4.34) also amounts to a one-to-one correspondence between sheaves and canonical presheaves. Hence we see that every sheaf is the sheaf of germs of some presheaf.

4.3. The Kostant Version of Supermanifolds. Let M^p (or simply M) be a C^∞ manifold of dimension p. Denote by $C^\infty(M)$ the canonical presheaf of smooth $\mathbb{R}$-valued functions on M. We now impose a $\mathbb{Z}_2$-grading (i.e., "super") on $C^\infty(M)$ in the trivial way: every element in the commutative $\mathbb{R}$-algebra $C^\infty(U)$ is even (i.e., grade 0).

Let $\mathscr{A}$ be a canonical presheaf of supercommutative algebras (with a unit element) on M such that $\mathscr{A}$ is presheaf-homomorphic to $C^\infty(M)$, i.e., if A_U is the stalk over U (an open set in M), then

$$\begin{array}{ccc} A_V & \xrightarrow{h_V} & C^\infty(V) \\ {\scriptstyle \rho_{UV}}\downarrow & \text{comm} & \downarrow{\scriptstyle \rho'_{UV}} \\ A_U & \xrightarrow[h_U]{} & C^\infty(U) \end{array} \tag{4.36}$$

where $\{h_U\}_U$ is the collection of algebra-homs that forms the presheaf-hom. The grading in A_U will be written

$$A_U = A_{U0} \oplus A_{U1} \tag{4.37}$$

where A_{U0} and A_{U1} denote respectively the even and the odd graded parts of A_U. To avoid confusing "odd elements" (which means elements of A_{U1}) with "odd number of elements", we use a hyphen as indicated if necessary (though odd is seldom used in the latter sense in this text). For convenience, we denote the image of the *presheaf-hom*, from $\mathscr{A}$ to $C^\infty(M)$, by a tilde.

$$h_U\colon a \mapsto a^\sim \quad \text{for } a \in A_U \tag{4.38}$$

Hence, clearly, due to $\mathbb{Z}_2$-grading we have

$$(A_{U1})^\sim = \{0\} \tag{4.39}$$

We need the notion of "factors" of A_U: A subalgebra of A_{U0} with 1_U (the unit of A_U) is called an **even-factor** (of A_U) if it is algebraically isomorphic to $C^\infty(U)$. A set (of odd-elements) $a_i \in A_{U1}$ ($i = 1, \ldots, n$) is **algebraically independent** if $a_1 \cdots a_n \neq 0$. Then a subalgebra of A_U is called an **odd-factor** (of **odd-dimension** q) if it is almost generated by a set of q algebraically independent odd-elements ("almost generate" means "together with 1_U they generate").

We note that if D_U is an odd-factor of odd-dim q, then the algebraic dim is obviously

$$\dim D_U = 2^q \tag{4.40}$$

since the generators (besides $1_U \in A_{U0}$) are all odd-elements.

The notion of "factor" will be introduced now: for a nonempty open U, a pair of subalgebras $\{E_U, D_U\}$ is a **splitting factor** of A_U (or $\{E_U, D_U\}$ **splits** A_U) if E_U and D_U are respectively even- and odd-factors of A_U such that

$$a \otimes b \mapsto ab \tag{4.41}$$

defines an algebra-iso

$$E_U \otimes D_U \leftrightarrow A_U \tag{4.42}$$

An open set U (of M) is said to be $\mathscr{A}$**-splitting** (or simply U splits) if A_U has a splitting factor. The **odd-dimension** of U is defined to be that of the odd-factor, i.e., if $\{E_U, D_U\}$ splits A_U with $\dim D_U = 2^q$ then odd-dim $U = q$ by definition. It is obvious that q is a unique positive integer if U splits. We now define $(M^p, \mathscr{A})$, where $\mathscr{A}$ is a canonical presheaf defined by (4.36), as a supermanifold of dimension (p, q) if every nonempty open set is covered by some $\mathscr{A}$-splitting open sets of the same odd-dim q. We also say that the supermanifold has even-dimension p and odd-dimension q.

We can immediately define the tangent (vector) and the tangent space of a supermanifold formally. Let $A \equiv A_M$ (i.e., let $U \equiv M$ which is open by definition) and $(\ )^*$ be the dual space. Then $u \in A^*$ is said to be a **(super) tangent** of A at $x \in M$ if

$$u'(ab) = (u'a)\cdot\tilde{b}(x) + (-1)^{|a|}\tilde{a}(x)\cdot u'(b) \tag{4.43}$$

and

$$u''(ab) = (u''a)\cdot\tilde{b}(x) + \tilde{a}(x)\cdot u''(b) \tag{4.44}$$

where $u = u' + u''$ (with $|u''| = 0$, $|u'| = 1$) and $a, b \in A$. The **tangent space** $T(M, \mathscr{A})_x$ at $x \in M$ of the supermanifold $(M, \mathscr{A})$ is the space of all (super) tangents of A at x. It is not difficult to see that

$$(T(M, \mathscr{A})_x)_0 \leftrightarrow T(M)_x \tag{4.45}$$

and

$$\dim T(M, \mathscr{A})_x = \dim (T(M, \mathscr{A})_x)_0 + \dim (T(M, \mathscr{A})_x)_1 = p + q \tag{4.46}$$

An important aspect of a "manifold" is the coordinate system. For a supermanifold, the definition of its even part is straightforward: let U be open in M and let $r_i \in A_{U0}$ $(i = 1, \ldots, p)$. Then $\{r_i\}$ is an **even-coordinate**

system (ECS) in U if U is an ordinary coordinate patch and the functions $\{r_i^{\sim}\}$ form the coordinate maps of the ordinary coordinate system (ordinary = in the usual differential geometry). We list the following properties of an ECS:

1. If $\{r_i\}$ is an ECS in U, then there exists a unique even-factor C_U containing $\{r_i\}$.
2. U admits ECS $\{r_i\}$ iff U is an ordinary coordinate patch.
3. If $\{f_i\}, f_i \in C^\infty(U), i = 1, \ldots, p$, form a coordinate system of functions, then there exist even elements $\{r_i\}$ such that $r_i^{\sim} = f_i$.

To define the so-called odd coordinate systems (OCS) requires a different approach. From the discussion of a splitting factor, one can expect that an OCS must be tied up with q algebraically independent odd-elements. This feeling will be confirmed. The passage from A_{U0} to $C^\infty(U)$, for ECS, is uneventful due to (4.36) and (4.39), which make the same approach impractical for OCS. However, we can use the notion of sections of fibre-bundles; we are particularly interested in defining an equivalent class assigned to a point $x \in M$, like the germs given in (4.17) in terms of (4.15). The union in (4.15) is a disjoint one. Hence our construct should conform to this fact. Since an OCS deals with odd-elements and since $a^2 = 0$ for any odd-element a in A_U, we see that A_{U1} must be nilpotent (of nilpotency index 2). Thus the set

$$N_U \equiv \text{all nilpotent elements in } A_U \tag{4.47}$$

plays an important part in the process. N_U is clearly a *graded* (*two-sided*) *ideal* of A_U; it contains A_{U1} and has the following useful properties:

1. N_U is the ideal generated by A_{U1} if U splits.
2. If a supermanifold has odd-dim q, then for any nonempty open U

$$(N_U)^q \neq 0, \qquad (N_U)^{q+1} = 0 \tag{4.48}$$

3. For any open U

$$0 \longrightarrow N_U \xrightarrow{\text{inc.}} A_U \xrightarrow{h_U} C^\infty(U) \longrightarrow 0 \tag{4.49}$$

is an exact sequence.

4. If E_U is an even-factor of A_U, then

$$A_U = N_U \ddot{\oplus} E_U \qquad \text{(semidirect sum)} \tag{4.50}$$

We next define

$$Z_{U,x} \equiv \{f \mid f \in C^\infty(U), f(x) = 0\}, \qquad Z_x \equiv Z_{M,x} \tag{4.51}$$

$$\bar{N}_U{}^j \equiv (N_U)^j/(N_U)^{j+1} \tag{4.52}$$

Equation (4.52) is defined to establish the "disjointness" of different $\bar{N}_U{}^j$

($j = 1, \ldots, q$). Identify A_U/N_U with $C^\infty(U)$. Then the $\bar{N}_U{}^j$ form a module over $C^\infty(U)$. Denote

$$F_x{}^j \equiv \bar{N}^j/Z_x \cdot \bar{N}^j \quad \text{with } N^j \equiv N_M{}^j \tag{4.53}$$

and

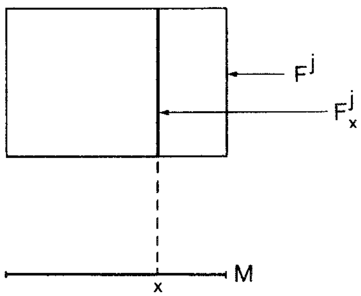
$$F^j \equiv \bigcup_{x \in M} F_x{}^j \tag{4.54}$$

In particular, if $x \in U$, we have

$$F_x{}^j \leftrightarrow \bar{N}_U{}^j/Z_{U,x} \cdot \bar{N}_U{}^j, \qquad j = 1, \ldots, q \tag{4.55}$$

Then $\{F^j, \pi, M\}$ is a smooth $\mathbb{R}$-line bundle with $F_x{}^j$ being the fibre at x. Clearly

$$\dim F_x{}^j = {}_qC_j \quad \text{(algebraic)} \tag{4.56}$$



(4.57)

Define the quotient map

$$(N_U)^j \to (N_U)^j/(N_U)^{j+1} \tag{4.58}$$

which induces the map

$$\tau_j : (N_U)^j \to \Gamma(U, F^j) \tag{4.59}$$

where $\Gamma(U, F^j)$ is the set of all sections over U in the fibre bundle F^j. Then we define an OCS (**odd-coordinate system**) as a set of odd-elements $\{a_i\}$, $i = 1, \ldots, q$, in A_U such that $\tau_q(a_1 \cdots a_q) \neq 0$ for all $x \in U$, i.e., τ yields a nowhere-vanishing section of F^q, over U. This leads immediately to the fact that the algebra almost generated by an OCS is an odd-factor of A_U. It can be shown [19] that if U splits and if q odd-elements q_i almost generate the odd-factor, then $\{a_i\}$ is an OCS in U. Hence OCS exists at least locally.

4.4. Diagrammatic Definitions of Some Algebraic Structures. The advantage of introducing algebraic definitions by commutative diagrams [21] is that "duality" can be established in a most direct and visible way, i.e., in the following diagram the concepts on the left-hand side induce those on the right-hand side and vice versa

$$\text{multiplication} \leftrightarrow \text{comultiplication}$$

$$\text{unit} \leftrightarrow \text{counit}$$

$$\text{associativity} \leftrightarrow \text{coassociativity}$$

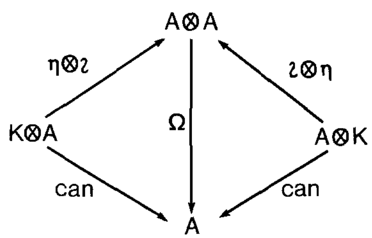
$$\text{algebra} \leftrightarrow \text{coalgebra}$$

and so on. We need these co-notions to define efficiently objects like Hopf algebra; they will be used, in turn, to define a super Lie group. In what follows, all diagrams are commutative. K denotes the ground field and also a trivial K-algebra. A and C are K-modules. ι is the identity map in different contexts. All the mappings are assumed to be K-linear in this section. Define

$$\Omega: A \otimes A \to A \quad (\textit{algebraic multiplication} \text{ on } A) \tag{4.60}$$

$$\eta: K \to A \quad (\text{the } \textit{unit} \text{ of } A) \tag{4.61}$$

Then A is an **algebra** with a *unit* when equipped with Ω and η such that



(4.62)

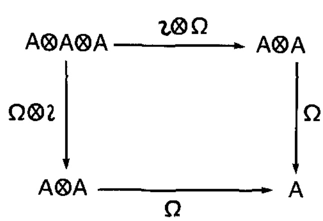
where can denotes the two canonical mappings:

$$\kappa \otimes a \mapsto \kappa a \tag{4.63}$$

and

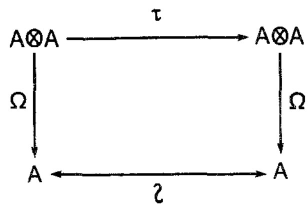
$$a \otimes \kappa \mapsto \kappa a, \qquad a \in A, \kappa \in K \tag{4.64}$$

A **associative** if



(4.65)

and **commutative** if



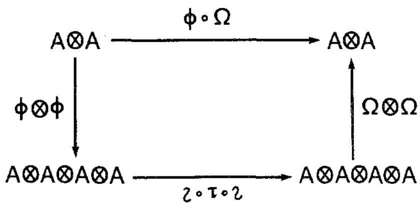
(4.66)

where τ is the **twist-morphism** defined by (for graded algebras; ungraded algebras to be considered zero-graded)

$$\tau: a \otimes b \mapsto (-1)^{|a||b|} b \otimes a \quad \text{with } a, b \in A \tag{4.67}$$

where a and b are homogeneous (nonhomogeneous elements are handled by their homogeneous components).

A **morphism** (of K-algebras) ϕ is a mapping such that



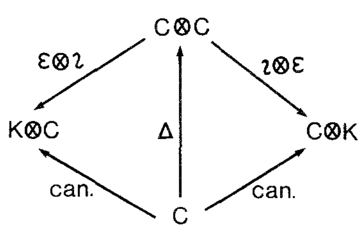
(4.68)

The dual notions are defined by simply reversing all arrows in the preceding diagrams; all co-objects are introduced in this way. Define first

$$\Delta: C \to C \otimes C \quad (\textit{comultiplication on } C) \tag{4.69}$$

$$\epsilon: C \to K \quad (\textit{counit of } C) \tag{4.70}$$

ϵ is also (most frequently) called an *augmentation of C*. Then C is a **coalgebra** with a *counit* when equipped with Δ and ϵ such that



(4.71)

where can are defined by

$$\kappa c \mapsto \kappa \otimes c \tag{4.72}$$

$$\kappa c \mapsto c \otimes \kappa, \qquad c \in C, \kappa \in K \tag{4.73}$$

Then a **Hopf algebra** is both an algebra and a coalgebra such that the comultiplication is a morphism, i.e., H is a Hopf algebra if $\{H, \Omega, \eta\}$ is an algebra and $\{H, \Delta, \epsilon\}$ is a coalgebra such that Δ is a morphism in the sense of (4.68). Algebras and coalgebras can be graded in the same way. Hence a Hopf algebra can have a graded structure in this sense. If A is a graded algebra or coalgebra and if $a, b \in A$ are such that

$$a \cdot b = (-1)^{|a||b|} b \cdot a \tag{4.74}$$

then we say that a and b **commute** in A. This graded commutativity (4.74) is just the one defined by (4.66) and (4.67).

In a coalgebra $\{C, \Delta, \epsilon\}$, the so-called antipodal map plays a useful role in many cases. Let $\alpha \in \mathrm{End}_K\, C$. Then α is an **antipodal map** if for any $x \in C$ and $\Delta x \equiv \sum_i x_i' \otimes x_i''$ we have

$$\sum_i \alpha(x_i') \cdot x_i'' = \sum x_i' \cdot \alpha(x_i'') = \epsilon(x) \tag{4.75}$$

Further, if C is a super coalgebra, then a nonzero $g \in C_0$ is **grouplike** if

$$\Delta : g \mapsto g \otimes g \tag{4.76}$$

and we say that $a \in C$ is **primitive** to g if g is grouplike and if

$$\Delta : a \mapsto g \otimes a + a \otimes g \tag{4.77}$$

These definitions are convenient for our discussion on super Lie groups in the next section.

4.5. Super Lie Groups and Super Lie Algebras. For two supermanifolds $(M, \mathscr{A})$ and $(M', \mathscr{A}')$, a **morphism** σ is defined in terms of the (commutative) superalgebras as

$$\sigma_* : A'_{M'} \xrightarrow[\text{hom}]{} A_M \quad \text{(superalgebra hom)} \tag{4.78}$$

For technical reasons, we shall pay special attention to the subspace $A^\dagger$ (of A^*, the dual of A) of all elements that annihilate some ideal, of A, of a finite codimension.

The mapping $f \mapsto (h_U f)x$, for any $f \in A$ and $x \in M$, defines an element γ_x in A^*:

$$\gamma_x : A \to \mathbb{R} \tag{4.79}$$

This permits us to attach $A^\dagger$ to a point $x \in M$ in the following way: for any nonnegative integer κ, define

$$A_x^{\kappa\dagger} \equiv \{v \mid v \in A^*, v \text{ annihilates } (\ker \gamma_x)^{\kappa+1}\} \tag{4.80}$$

and

$$A_x^\dagger \equiv \bigcup_\kappa A_x^{\kappa\dagger} \tag{4.81}$$

It is clear that $A_x{}^\dagger$ is contained in $A_x^{\kappa+1\dagger}$. One of the important relations is

$$A^\dagger = \bigoplus_{x \in M} A_x^\dagger \tag{4.82}$$

where $A_x^\dagger$ is obviously a sub "super" coalgebra of $A^\dagger$. In fact, according to Kostant, each $A_x^\dagger$ has a unique grouplike element γ_x [defined by (4.79)] and

$$T(M, \mathscr{A})_x \equiv \{\tau | \tau \in A_x^\dagger,\ \tau \text{ is } \textit{primitive} \text{ to } \gamma_x\} \tag{4.83}$$

i.e., the tangent space of the supermanifold at a point x is just the set of all elements primitive to γ_x. This statement of Kostant is a very elegant one. Because of the uniqueness of the grouplike elements in this case, the correspondence $x \mapsto \gamma_x$ enables us to identify *injectively* M as a subset of $A^\dagger$, i.e., we may identify M as the grouplike elements in $A^\dagger$.

Let σ be a morphism from supermanifolds $(M, \mathscr{A})$ to $(M', \mathscr{A}')$, i.e.,

$$\sigma^\dagger : A' \xrightarrow[\text{hom}]{} A \tag{4.84}$$

The induced map

$$\sigma^\dagger : A^\dagger \to A'^\dagger \tag{4.85}$$

defined by

$$\sigma^\dagger : v \mapsto v \circ \sigma_* \tag{4.86}$$

is a super coalgebra morphism. In fact, the grouplikeness is preserved under $\sigma^\dagger$, i.e.,

$$\sigma^\dagger : M \to M' \tag{4.87}$$

Based on (4.85), we define: a super coalgebra morphism ψ from $A^\dagger$ to $A'^\dagger$ is said to be **smooth** if

$$\exists \text{ supermanifold morphism } \sigma_* : A' \to A \quad \text{such that } \psi = \sigma_* \tag{4.88}$$

Then a supermanifold $(M, \mathscr{A})$ is a **super Lie group** (SLG) if $A^\dagger$ is a graded Hopf algebra with a smooth multiplication and a smooth antipodal map. The *antipodal map* is defined specifically by

$$A^\dagger \to A^\dagger$$

with

$$x \mapsto -x \tag{4.89}$$

An immediate consequence of the definition of SLG is this: if $(G, \mathscr{A})$ is an SLG, then G is a group since its elements are grouplike and since there is an antipodal map. In fact G is an ordinary Lie group. Denote the unit element of G (as a group) by e. Then $T(G, \mathscr{A})_e \equiv \mathscr{I}$ is just $A^\dagger$ (i.e., $A_G^\dagger$), consisting of all elements primitive to e. $\mathscr{I}$ is an SLA (super Lie algebra), to be called the **Lie algebra of the SLG** $(G, \mathscr{A})$. Decompose

$$\mathscr{I} = \mathscr{I}_0 + \mathscr{I}_1 \tag{4.90}$$

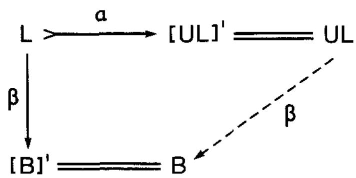
Then the even part $\mathscr{I}_0 = T(G)_e$ is just the ordinary Lie algebra of a Lie group G.

Universal Enveloping Algebra of a Super Lie Algebra. Universal enveloping algebra is a useful tool for developing the representation theory of a Lie algebra. The technique can be easily extended to the case of an SLA without undue complications.

Let L be an SLA over K with

$$L = L_0 \oplus L_1 \quad \text{(even and odd parts)} \tag{4.91}$$

Let B be any SAA (super associative algebra) over K, and denote by $[B]'$ the SLA of B with multiplication defined by (4.11). Let β be any SLA-hom from L to $[B]'$. Then the **universal enveloping algebra** (UEA) of L, to be denoted by UL, is defined as an SAA with an SLA-hom α from L to $[UL]'$ such that the dotted arrow in the following diagram is filled by a unique SAA-hom β'.



(4.92)

Hence, as for an ordinary Lie algebra, UL is unique (up to an SAA-iso) for an SLA L. The construction of UL is similar to that of an ordinary Lie algebra.

$$UL = TL \bmod JL \tag{4.93}$$

where $TL = \bigoplus_{i=0}^{\infty} \otimes^i L$ and JL is the ideal in TL generated by elements

$$x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y]', \qquad x, y \in L \tag{4.94}$$

UL as a SAA has even and odd parts

$$UL = (UL)_0 \oplus (UL)_1 \tag{4.95}$$

The usual Poincaré–Birkhoff–Witt theorem is also true in this case [12], i.e., the set of all distinct lexicographically ordered monomials $\overline{x^\nu y^\mu}$, i.e. (written in full)

$$x_1^{\nu_1} \otimes \cdots \otimes x_p^{\nu_p} \otimes y_1^{\mu_1} \otimes \cdots \otimes y_q^{\mu_q} \bmod JL \tag{4.96}$$

form a base of UL, where $\{x_1, \ldots, x_p\}$ is a base of L_0 and $\{y_1, \ldots, y_q\}$ a base of L_1. The appearance of x_i° and y_j° means that x_i and x_j are absent in that monomial. Actually $\mu_i = 0$ or 1 since y_i are odd-elements. Further, whether a monomial $\overline{x^\nu y^\mu}$ is even or odd depends only on the length of μ (i.e.,

$|\mu_1| + \cdots + |\mu_q|$). Denote by UL_0 the sub-SAA (in UL) generated by $Ke \oplus L_0$ under the quotient (i.e., mod JL), where e is the 1 of UL. Denote by VL_0 and VL_1 the K-subspaces (in UL) generated by e and monomials in y_i of even and odd lengths, respectively. Then we have

$$(UL)_0 = VL_0 \otimes_K UL_0 \tag{4.97}$$

and

$$(UL)_1 = VL_1 \otimes_K UL_0 \tag{4.98}$$

The **trace form** (sometimes referred to as the **supertrace form** of an SLA L, with respect to a given representation γ, is defined by

$$\langle a, b\rangle_\gamma \equiv \langle a, b\rangle_{\gamma,0} - \langle a, b\rangle_{\gamma,1} \tag{4.99}$$

where $\langle a, b\rangle_{\gamma,i} \equiv \mathrm{Tr}\,(\gamma(a)|_{V_i}\gamma(b)|_{V_i})$, $i = 0, 1$, and V is the L-module corresponding to γ. $(\ )|_{V_i}$ denotes the restriction to the subspace V_i. Then the **(super) Killing form** is just the trace form with respect to the adjoint representation, to be denoted simply by $\langle\ ,\ \rangle$. It is easy to check that a trace form is L-invariant, i.e.,

$$\langle [a, c]', b\rangle_\gamma = \langle a, [c, b]'\rangle_\gamma \tag{4.100}$$

or

$$\langle [c, a]', b\rangle_\gamma + (-1)^{|c||a|}\langle a, [c, b]'\rangle_\gamma = 0 \tag{4.101}$$

Besides, a trace form is "super" symmetric, i.e.,

$$\langle a, b\rangle_\gamma = (-1)^{|a||b|}\langle b, a\rangle_\gamma \tag{4.102}$$

If the supertrace (with respect to a certain representation γ) is *nondegenerate* on an SLA L then we can define the second-order Casimir element by

$$\sum_{i=1}^{p+q} (-1)^{|b_i|} b_i \otimes b_i^* \tag{4.103}$$

where $b_1, \ldots, b_{p+q}$ is a base of L and $\{b_i^*\}$ is the dual base, i.e.,

$$\langle b_i, b_j^*\rangle_\gamma = \delta_{ij} \tag{4.104}$$

There are two different definitions of semisimplicity. We shall consider the following definition [15]: An SLA is **semisimple** if it has no nontrivial solvable ideal. It is **simple** if it is not abelian and has no nontrivial ideal. Kac gave a classification [15, 22] of simple SLAs over the fields $\mathbb{R}$ and $\mathbb{C}$. If the base $\{b_i\}_{p+q}$ is such that $\{b_i\}_{1,\ldots,p}$ spans L_0 and $\{b_i\}_{p+1,\ldots,q}$ spans L_1, then (4.103) becomes simply

$$\sum_{i=1}^{p} b_i \otimes b_i^* - \sum_{j=p+1}^{p+q} b_j \otimes b_j^* \tag{4.105}$$

More generally, in a representation γ, denote the **supertrace metric** by

$$g_{ij} \equiv \langle b_i, b_j\rangle_\gamma \tag{4.106}$$

If det $\|g_{ij}\| \neq 0$, then the *jth order Casimir operator* can be constructed in a way analogous to the ordinary Lie algebra, i.e.,

$$\Gamma_j \equiv \sum_{i_1,\ldots,i_j=1}^{n} \operatorname{str}(b_{i_1}^{\gamma} \cdots b_{i_j}^{\gamma}) \cdot b^{i_1} \otimes \cdots \otimes b^{i_j} \bmod JL \tag{4.107}$$

where, in particular, γ may be the adjoint representation and indices of b_i are raised by the Killing metric, e.g.,

$$b^k \equiv \sum_{m=1}^{n} g^{km} b_m \qquad (n = p + q) \tag{4.108}$$

and str is the supertrace, defined by

$$\operatorname{str}(\ ) \equiv \operatorname{Tr}(\text{even part}) - \operatorname{Tr}(\text{odd part}) \tag{4.109}$$

where Tr (even part) means restrictions to the even part of the representation space. It can be shown that the Casimir elements commute with every element of the SLA (i.e., they belong to the center of the SLA). However, their eigenvalues do not specify the representation for a simple SLA in general. Yet the irreducible representation of a simple SLA can be specified by the highest weight. It is also true that the Killing form for a simple SLA is not necessarily nondegenerate; but if that is the case, then the simple SLA is of "classical" type. Though Engel's theorem is still valid for an SLA, Lie's theorem for a solvable LA cannot be extended to a solvable SLA. Weyl's theorem and Levi-Malcev's theorem are not true for the SLA case.

5. COHOMOLOGIES OF SUPERMANIFOLDS AND SUPER LIE ALGEBRAS

5.1. Cohomology of Supermanifolds. For a supermanifold $(M, \mathscr{A})$ and for any open $U \subset M$, Der A_U has an SLA structure and a free A_U-module. In fact

$$U \mapsto \operatorname{Der} A_U \tag{5.1}$$

defines a canonical presheaf of A_U-modules; the module structure is, in fact, compatible with the restriction maps. In particular, if U splits, say $\{E_U, D_U\}$ splits A_U, then Der A_U has a decomposition into a vector space direct sum.

$$\operatorname{Der} A_U = \operatorname{Der}(A_U|E_U) + \operatorname{Der}(A_U|D_U) \tag{5.2}$$

with

$$\operatorname{Der}(A_U|E_U) \equiv \{\zeta | \zeta \in \operatorname{Der} A_U, \zeta(E_U) = 0\} \tag{5.3}$$

and a similar definition for Der $(A_U|D_U)$.

At this stage it is convenient to introduce a further notion of coordinate neighborhood: a (ordinary) coordinate neighborhood U is an $\mathscr{A}$-coordinate neighborhood if A_U has an OCS. $\{r_i, s_j\}_{i=1,\ldots,p,j=1,\ldots,q}$, is an $\mathscr{A}$-coordinate system if $\{r_i\}$ and $\{s_j\}$ are ECS and OCS, respectively. It can be shown [19] that if U is an $\mathscr{A}$-coordinate neighborhood with an $\mathscr{A}$-coordinate system $\{r_i, s_j\}$, then there *exist uniquely*

$$\partial/\partial r_i, \partial/\partial s_j \in \operatorname{Der} A_U \qquad (i = 1, \ldots, p; j = 1, \ldots, q) \tag{5.4}$$

such that

$$(\partial/\partial r_i)r_k = 1_U\delta_{ik}, \qquad (\partial/\partial s_j)s_k = 1_U\delta_{jk} \tag{5.5}$$

$$(\partial/\partial r_i)s_j = 0, \qquad (\partial/\partial s_j)r_i = 0 \tag{5.6}$$

Clearly, their homogeneous degrees are

$$|\partial/\partial r_i| = 0 \quad \text{and} \quad |\partial/\partial s_j| = 1 \tag{5.7}$$

Thus every $\zeta \in \operatorname{Der} A_U$ has a unique expression

$$\zeta = \sum_{i=1}^{p} a_i\partial/\partial r_i + \sum_{j=1}^{q} b_i\partial/\partial s_j, \qquad a_i, b_j \in A_U \tag{5.8}$$

which also corresponds to the decomposition (5.2), i.e., Der A_U is a *free* A_U-module with a base given by (5.4). For each U we can now construct the tensor algebra

$$T_U \equiv \sum_{t=0}^{\infty} \bigotimes_{AU}^{t} \operatorname{Der} A_U \tag{5.9}$$

which is *bigraded* by $\mathbb{Z} \oplus \mathbb{Z}_2$ (the "tensorial" grading is $\mathbb{Z}$). Denote

$$T_U^{(t)} \equiv \bigotimes_{AU}^{t} \operatorname{Der} A_U \tag{5.10}$$

$J_U \equiv$ two-sided bigraded ideal in T_U generated by elements of the form $\zeta \otimes \zeta' + (-1)^{|\zeta||\zeta'|}\zeta' \otimes \zeta$ with $\zeta, \zeta' \in \operatorname{Der} A_U$ being homogeneous elements.

$$J_U^{(t)} \equiv J_U \cap T_U^{(t)} \tag{5.11}$$

Both $T_U^{(t)}$ and A_U itself have the structure of left A_U-modules. Denote $T_U^{(t)\star} \equiv \operatorname{Hom}_{AU}(T_U^{(t)}, A_U)$ and similarly for $J_U^{(t)\star}$. Then for any $\alpha \in T_U^{(t)\star}$,

$$\alpha(\zeta_1, \ldots, a\cdot\zeta_k, \ldots, \zeta_t) = (-1)^{|\alpha|\sum_{i=1}^{k-1}|\zeta_i|}f\cdot(\alpha(\zeta_1, \ldots, \zeta_k, \ldots, \zeta_t)) \tag{5.12}$$

where $\zeta_i \in \operatorname{Der} A_U$ and $a \in A_U$. Let

$$\mathscr{U}_U^{(t)} \equiv \{\beta | \beta \in T_U^{(t)\star}, \ker\beta \supset J_U^{(t)}\} \tag{5.13}$$

and

$$\mathscr{U}_U \equiv \bigoplus_{t=0}^{\infty} \mathscr{U}_U^{(t)} \quad \text{with } \mathscr{U}_U{}^{\circ} \equiv A_U \tag{5.14}$$

$\mathscr{U}_U$ is a $(\mathbb{Z} \oplus \mathbb{Z}_2)$-graded commutative algebra over A_U. In other words, if

$$\beta \in (\mathscr{U}_U^{(t)})_i \quad \text{and} \quad \beta' \in (\mathscr{U}^{(t')})_{i'} \tag{5.15}$$

with $i, i' \in \mathbb{Z}_2$ and $t, t' \in \mathbb{Z}_+ \subset \mathbb{Z}$, then

$$\beta\beta' = (-1)^{tt'+ii'}\beta'\beta \in (\mathscr{U}_U^{(t+t')})_{i+i'} \tag{5.16}$$

where the subscript i in (5.15) refers to the ith supercomponent.

The *exterior derivative* can be defined by

$$d: \mathscr{U}_U^{(0)} \to \mathscr{U}_U^{(1)} \tag{5.17}$$

with

$$(da)\zeta \equiv \zeta a \quad \text{for any } \zeta \in \operatorname{Der} A_U \tag{5.18}$$

In terms of an $\mathscr{A}$-coordinate system (5.4), we have

$$d = \sum_i (dr_i)\cdot(\partial/\partial r_i) + \sum_j (ds_j)\cdot(\partial/\partial s_j) \tag{5.19}$$

It is not difficult to show that for any $a, b \in A_U$

$$d(a\cdot b) = (da)\cdot b + a\cdot(db) \tag{5.20}$$

which does *not* have any sign factor in the second term as in the ordinary differentiable manifold theory. Further, we note that

$$dr_i\cdot ds_j = -ds_j\cdot dr_i$$

$$(\partial/\partial r_i)\cdot(\partial/\partial s_j) = (\partial/\partial s_j)\cdot(\partial/\partial r_i)$$

$$(\partial/\partial s_j)\cdot(\partial/\partial s_k) = -(\partial/\partial s_k)\cdot(\partial/\partial s_j)$$

because the (super) degrees are

$$|\partial/\partial r_i| = 0 \quad \text{and} \quad |\partial/\partial s_j| = 1 \tag{5.21}$$

The *interior product* i_ζ is defined as follows: let $\zeta \in \operatorname{Der} A_U$ be homogeneous. Then we define

$$i_\zeta: \mathscr{U}_U^{(t+1)} \to \mathscr{U}_U^{(t)}$$

by

$$(i_\zeta u)(\zeta_1, \ldots, \zeta_t) = (-1)^{|\zeta|\sum_{i=1}^t |\zeta_i|} u(\zeta, \zeta_1, \ldots, \zeta_t) \tag{5.22}$$

for any $u \in \mathscr{U}_U^{(t+1)}$. The following facts are obvious.

$$\mathbb{Z}\text{-degree of } i_\zeta = -1, \qquad |i_\zeta| = |\zeta| \tag{5.23}$$

$$(i_{a\cdot\zeta})u = a\cdot(i_\zeta u) \tag{5.24}$$

with $\zeta \in \operatorname{Der} A_U$, $u \in \mathscr{U}_U$, $a \in A_U$. The *Lie derivative* is defined by

$$\pounds_\zeta \equiv d \circ i_\zeta + i_\zeta \circ d \tag{5.25}$$

which leads to

$$d \circ \pounds_\zeta = \pounds_\zeta \circ d \tag{5.26}$$

The Poincaré lemma holds in a supermanifold: let U be a connected contractible $\mathscr{A}$-coordinate neighborhood on M. If $u \in \mathscr{U}_U^{(t)}$ is closed then it is *exact*, i.e., $du = 0$ implies $u = dv$ for some v in $\mathscr{U}_U^{(t-1)}$. Denote by $\mathscr{U}^{(t)}(\mathscr{A})$ the canonical presheaf

$$U \mapsto \mathscr{U}_U^{(t)} \tag{5.27}$$

Then the exterior derivative yields a "flabby resolution" of the contrast sheaf [23]

$$\cdots \to \mathscr{U}^{(t)}(\mathscr{A}) \xrightarrow{d} \mathscr{U}^{(t+1)}(\mathscr{A}) \to \cdots \tag{5.28}$$

We introduce here the following notation:

$H(M, \mathbb{R})$: Cohomology of an ordinary manifold with coefficients in $\mathbb{R}$

$\overline{H}(\mathbf{C})$: Cohomology on a cochain complex $\mathbf{C}$ with given coboundary operators

$\check{H}(M, \mathscr{A})$: Čech cohomology with values in $\mathscr{A}$

$H_{\text{Rham}}(M, \mathscr{A})$: de Rham cohomology of $(M, \mathscr{A})$ defined by

$$\overline{H}(M, \mathscr{A}) = H(\mathscr{U}_M(\mathscr{A})) \tag{5.29}$$

By de Rham's theorem (applied to (5.28)), we have an isomorphism

$$H_{\text{Rham}}(M, \mathscr{A}) \leftrightarrow H(M, \mathbb{R}) \leftrightarrow H_{\text{Rham}}(M) \tag{5.30}$$

5.2. Cohomology of Super Lie Algebras. In the case of an ordinary Lie algebra L, it is advantageous to consider the cohomology with coefficients in an L-module (i.e., a representation space of L). See, for example, [6], p. 856. We shall also take this approach for SLAs. Let L be an SLA over K. If V is an L-supermodule, then

$$V = V_0 \dotplus V_1 \tag{5.31}$$

Denote by $F_m(L, V)$ the set of all K-multilinear mappings from $\otimes^m L$ into V. Then $\omega \in F_m(L, V)$ is an **exterior m-form** if

$$\omega(\ldots, x_i, x_{i+1}, \ldots) = -(-1)^{|x_i||x_{i+1}|}\omega(\ldots, x_{i+1}, x_i, \ldots) \tag{5.32}$$

Let $\Lambda_m(L, V)$ be the set of all exterior m-forms. Denote

$$\Lambda(L, V) \equiv \bigoplus_{m=0}^{\infty} \Lambda_m(L, V) \tag{5.33}$$

Λ is a superspace. We now proceed to define the *exterior derivative* on Λ

$$d\colon \Lambda_m \to \Lambda_{m+1} \tag{5.34}$$

For $m > 0$, we define

$$\begin{aligned}(d\omega)(x_1, \ldots, x_{m+1}) \equiv & \sum_{i=1}^{m+1} (-1)^{\alpha_i} x_i \cdot (\omega(x_1 \ldots, \hat{x}_i, \ldots, x_{m+1})) \\ & + \sum_{\substack{i \\ (i<j)}} \sum_{j} (-1)^{\beta_{ij}} \omega([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{m+1})\end{aligned} \tag{5.35}$$

where $\hat{x}_i$ means deletion of x_i and

$$\alpha_i \equiv (i - 1) + |x_i| \sum_{j=1}^{i-1} |x_j| \tag{5.36}$$

$$\beta_{ij} \equiv (i + j) + (|x_i| + |x_j|) \sum_{k=1}^{i-1} |x_k| + |x_j| \sum_{k=i+1}^{j-1} |x_k| \tag{5.37}$$

For $m = 0$, we identify $\Lambda_0(L, V)$ with V since the former sends $K1$ into V under a K-homomorphism. Hence we define d: $\Lambda_0 \to \Lambda_1$ by

$$(dv)x \equiv (-1)^{|x||v|} x \cdot v, \qquad v \in V, x \in L \tag{5.38}$$

The reason for defining α_i and β_{ij} as given in (5.36) and (5.37) is based on moving the ith term, e.g., to the leftmost position; each step involves a factor $(-1)^{|x_i||x_k|}$, etc.

The *tensor product* $\Lambda_m \otimes \Lambda_n$ is defined by

$$(\mu \otimes \nu)(x_1, \ldots, x_{m+n}) \equiv (-1)^{|\nu| \sum_{i=1}^m |x_i|} \mu(x_1, \ldots, x_m) \otimes \nu(x_{m+1}, \ldots, x_{m+n}) \tag{5.39}$$

The *exterior product* between Λ_m and Λ_n is defined by

$$\mu \wedge \nu \equiv \frac{1}{m!n!} \sum_{\mathbb{P}} \mathbb{P}(\mu \otimes \nu) \tag{5.40}$$

where $\mathbb{P}(\omega)$ is defined by

$$(\mathbb{P}\omega)(x_1, \ldots, x_k) = (-1)^{P^* + P} \omega(x_{\mathbb{P}^{-1}(1)}, \ldots, x_{\mathbb{P}^{-1}(m)}) \tag{5.41}$$

$\mathbb{P}$ is a *permutation* and $\mathbb{P}^*$ is induced by $\mathbb{P}$ only on the odd-elements of $(x_1, \ldots, x_k)$. P^* and P are signs of respective permutations.

The interior product, for $\omega \in \Lambda_m$, is defined by

$$i_x\omega(x_1, \ldots, x_{m-1}) \equiv (-1)^{|x||\omega|} \omega(x, x_1, \ldots, x_{m-1}) \quad \text{for } m > 0 \tag{5.42}$$

$$i_x v \equiv 0 \tag{5.43}$$

The Lie derivative is defined by

$$(\pounds_x\omega)(x_1, \ldots, x_m) \equiv x \cdot (\omega(x_1, \ldots, x_m)) - \sum_{i=1}^{m} (-1)^{\alpha_i} \omega(x_1, \ldots, [x, x_i], \ldots, x_m)$$

with

$$\alpha_i = |x| + \sum_{j=1}^{i-1} |x_j| \tag{5.44}$$

It is not difficult to verify the following properties:

$$\begin{aligned}
d^2 &= 0, \qquad \pounds_x = d \circ i_x + i_x \circ d \\
\mu \wedge \nu &= (-1)^{|\mu||\nu| + mn} \nu \wedge \mu \quad \text{if } \mu \in \Lambda_m, \nu \in \Lambda_n \\
d(\mu \wedge \nu) &= (d\mu) \wedge \nu + (-1)^m \mu \wedge (d\nu) \\
i_{a \cdot x}\omega &= a \cdot (i_x \omega) \\
(\omega \wedge \mu) \wedge \nu &= \omega \wedge (\mu \wedge \nu)
\end{aligned} \tag{5.45}$$

The cochain complex $\Lambda(L, V)$ together with the exterior derivatives (as coboundary operators) now defines the cohomology of the *SLA* L with coefficients in the L-module V, as in the usual construction of a cohomology theory. One of the results [24] is this: for a semisimple SLA L and a nontrivial simple L-module V with nondegenerate (super) trace form, the nth cohomology group $H^n(L, V)$ vanishes.

6. AN INSIGHT TO GENERALIZATIONS OF LIE ALGEBRAS

It is natural to ask how Lie algebra can be sensibly generalized beyond the super or $\mathbb{Z} \oplus \mathbb{Z}_2$ grading. I would like to proceed with an analysis of basic axioms involved. To avoid unnecessary confusion, I will now introduce a different notation. This is because the usual bracket [,] is inadequate on two accounts. First, a succession of brackets is inconvenient both to write as well as to read. Second, there is no distinction between the Lie product and the true commutator; the latter is defined only when an associative (or non-Lie) product makes sense. We now take $x \star y$ to be the Lie product in an ordinary Lie algebra, and $x \star' y$ for the super Lie algebra. [,] and [,]′ now represent the true bracket and graded bracket, $a \circ b - b \circ a$ and $a \circ b - (-1)^{|a||b|} b \circ a$, when they are well defined.

We first take a close look at the Jacobi identity for an ordinary Lie algebra L

$$x \star (y \star z) + y \star (z \star x) + z \star (x \star y) = 0 \tag{6.1}$$

and rewrite it in adjoint form

$$\{(\operatorname{adj} x) \circ (\operatorname{adj} y) - (\operatorname{adj} y) \circ (\operatorname{adj} x)\}z - (\operatorname{adj}(x \star y))z = 0 \tag{6.2}$$

or

$$[\operatorname{adj} x, \operatorname{adj} y] = \operatorname{adj}(x \star y) \tag{6.3}$$

since z is arbitrary and adjoints are endomorphisms on the vector space L. However, we can reverse the argument and use (6.3) as the axiom to replace the Jacobi identity (6.1) and also assume that on the left-hand side of (6.3) the "true" bracket is not yet defined (though the adjoint is defined in the conventional way). Then we see that the choice

$$[a, b] \equiv a \circ b - b \circ a \tag{6.4}$$

for a and b in an associative algebra, in general, reduces (6.3) to (6.1), the Jacobi identity. This game can also be played for a GLA. In this case, equation (3.6) in the new notation is

$$\sum_{\text{cyc}} (-1)^{|x||z|}(x \bigstar' y) \bigstar' z = 0 \tag{6.5}$$

which can be written

$$(\text{adj}\, x) \circ (\text{adj}\, y) = \text{adj}\,(x \bigstar' y) + (-1)^{|x||y|}(\text{adj}\, y) \circ (\text{adj}\, x) \tag{6.6}$$

or

$$[\text{adj}\, x, \text{adj}\, y]' = \text{adj}\,(x \bigstar' y) \tag{6.7}$$

In other words, we can replace (6.5) by (6.7). It is most striking that (6.3) and (6.7) take exactly the same form. This not only shows why GLA is so natural but also shows that "adjoint" respects the Lie product (or graded Lie product). This latter property is very crucial in defining morphisms, and representation theory (L-modules) depends on it. The burden of signs in (6.5) is buried in (6.7) without ceremony. We now take a further step. Take (6.7) as an axiom to replace the graded Jacobi identity, and let the left-hand side of (6.7) be undefined yet; it should be defined in a way that recovers the graded Jacobi identity.

The graded antisymmetry says

$$x \bigstar' y = -(-1)^{|x||y|} y \bigstar' x \tag{6.8}$$

To recover (6.5) from (6.7), we now define

$$[a, b]' = a \circ b - (-1)^{|a||b|} b \circ a \tag{6.9}$$

when a and b are elements in an associative (or some non-Lie) algebra such that the right-hand side of (6.9) makes sense.

We now proceed to consider a possible generalization of a Lie algebra. First, bilinearity of an algebraic product requires a *generalized antisymmetry* to look like (write the "new" product $a * b$)

$$x * y = f(x, y) y * x \tag{6.10}$$

where f maps $L \times L$ into K (L denotes the "generalized" Lie algebra over a field K of characteristic zero). Note that f is not required to be bilinear. An iteration of (6.10) shows that

$$f(x, y) f(y, x) = 1, \quad \text{for any } x, y \in L \tag{6.11}$$

must be satisfied. In particular, $(f(x, x))^2 = 1$. Since we do not know what the generalized Jacobi identity should look like, there is no reason why we cannot use (6.3) and (6.7) as guides, i.e., we require

$$[\text{adj } x, \text{adj } y]^* = \text{adj } (x * y) \tag{6.12}$$

and define

$$[a, b]^* \equiv a \circ b + f(a, b) b \circ a \tag{6.13}$$

imitating the previous cases. In this way, the "new" Jacobi identity looks like

$$x * (y * z) - (x * y) * z + f(x, y) y * (x * z) = 0 \tag{6.14}$$

Then, to narrow the generality, one can impose all kinds of conditions on f, as long as (6.11) is satisfied, to define different classes of new "generalized" Lie algebras. A recent attempt by V. Rittenberg and D. Wyler [25] is a special case of this formalism, done in terms of generators. Clearly, generators can be labeled by any index set. Then "gradings" of all sorts can be imposed to define a generalized graded Lie algebra. In fact, as we mentioned earlier in connection with the generalized (or signed) Whitehead product, grading conditions can be sensibly relaxed as given by equation (3.75).

ACKNOWLEDGMENTS

It is a pleasure to thank James Arnold, Georgia Benkart, Kuo-Tsai Chen, Bernice Durand, Y. Fujimoto, C. H. Sung, and H. S. Tsao for helpful discussions and their kind interest.

REFERENCES

1. Holstein, T., and Primakoff, H. (1940). *Phys. Rev.*, **58**, 1098.
2. Dyson, F. J. (1956). *Phys. Rev.*, **102**, 1217, 1230; Maleev, S. (1957). *Soviet Phys., JEPT*, **6**, 776.
3. Chow, Y., and Liu, C. J. (1973). *Int. J. Quan. Chem. Symp.*, **7**, 521.
4. Liu, C. J., and Chow, Y. (1971). *J. Math. Phys.*, **12**, 2144.
5. Weinberg, S. (1968). *Phys. Rev.*, **166**, 1568.
6. Chow, Y. (1978). *General Theory of Lie Algebras*, Gordon and Breach, New York.
7. Trautman, A. (1970). *Rep. on Math. Phys.*, **1**, 29; (1976), **10**, 297.
8. Yang, C. N. *Gauge Fields*, Proc. 1975 Hawaii Conference, University Press of Hawaii; Chern, S. S. (1977). In *Geometry and Topology*, ed. Palis et al., Lecture Notes in Math. No. 597, Springer-Verlag, New York, p. 114.
9. Kostant, B. (1970). In Lecture Notes in Math. No. 170, Springer-Verlag, New York, p. 87; Souriau, J. M. (1970). *Structure des systemes dynamiques*, Dunod; Simms, D. J., and Woodhouse, N. M. J. (1976). *Geometric Quantization*, Springer-Verlag, New York.
10. Hehl, F. W., Heyde, P., Kerlick, G. D., and Nester, J. M. (1976). *Rev. Mod. Phys.*, **48**, 393; Cho, Y. M. (1976). *Phys. Rev.*, **D14**, 3335, 3341.

11. Ehresmann, C. (1947). *Colloque de Topologie Algebraique*, CNRS, Paris, p. 3; (1950). *Coloque de Topologie*, Bruxelles, p. 29; Kobayashi, S. (1957). *Annali di Matematica*, **43**, 119.
12. Milnor, J. W., and Moore, J. C. (1965). *Ann. of Math.*, **81**, 211.
13. Corwin, L., Ne'eman, Y., and Sternberg, S. (1975). *Rev. Mod. Phys.*, **47**, 573.
14. Fayet, P., and Ferrara, S. (1977). *Phys. Report*, **32**, no. 5, 249; Ogievetskii, V. I., and Mezincescu, L. (1975). *Usp. Fiz. Nauk*, **117**, 637.
15. Kac, V. G. (1977). *Commun. Math. Phys.*, **53**, 31.
16. Berezin, F. A., and Leites, D. A. (1975). *Dokl. Akad. Nauk SSSR*, **224**, 505.
17. Berezin, F. A., and Kac, V. G. (1970). *Mat. Sbornik*, **82**, 343.
18. Lazard, M. (1955). *Ann. Sci. Ecole Norm. Sup.*, **72**, 299.
19. Kostant, B. (1977). Lecture Notes in Math. No. 570, Springer-Verlag, New York, p. 177.
20. Tennison, B. R. (1975). *Sheave Theory*, Cambridge University Press, Cambridge; Cartan, Henri. (1950/51). Seminaire H. Cartan, E.N.S., *Topologie algebrique, Faisceaux sur un espace topologique*; Chern, S. S. (1959). *Complex Manifolds*, Textos de Mat. No. 5, Recife, Brazil.
21. Sweedler, M. E. (1969). *Hopf Algebras*, W. A. Benjamin, New York.
22. Rittenberg, V. "A Guide to Lie Superalgebras," Rockefeller University Preprint, Report no. COO-2232B-141.
23. Sorani, G. (1969). *An Introduction to Real and Complex Manifolds*, Gordon and Breach, New York.
24. Leites, D. A. (1975). *Funktsional'nyi Analiz i Ego Prilozheniya*, **9**, 75.
25. Rittenberg, V., and Wyler, D. (1978). "Generalized Superalgebras," Rockefeller University Preprint, Report No. COO-2232B-150.